1 Problem 1

1.1 a. We have
\[ y'' + \lambda^2 x^4 y = 0, \quad y(1) = 0, \quad y'(1) = 1 \]
so that \( p(x) = \lambda x^2 \), and \( P(x) = x^2 \), so that a first-order approximation would have the form
\[ v_0 e^{\pm i \int \frac{p(t)}{x} dt} = 1 \]
with \( v_0 = \frac{1}{\sqrt{P(x)}} \). Note that \( v_0 \) is real, otherwise we would have had to take its complex conjugate to multiply the term with the minus sign in the exponent. We may thus rewrite this as
\[ y_0(x) = \frac{A}{x} e^{i\lambda \int \frac{t^3}{2} dt} + \frac{B}{x} e^{-i\lambda \int \frac{t^3}{2} dt} \]
Using the initial conditions, we find \( A \) and \( B \):
\[ y_0(1) = \frac{A + B}{1} = 0 \Rightarrow B = -A \]
and
\[ y'_0(1) = \frac{1}{1^2} \left( i\lambda 1^2 (A - B) \cdot 1 - (A + B) \cdot 1 \right) = 2iA\lambda = 1. \]
Thus we find \( A \):
\[ A = -i/2\lambda. \]
And the WKB approximation is
\[ y_0(x) = \frac{i}{2\lambda x} \left( -e^{i\lambda \int \frac{t^3}{2} dt} + e^{-i\lambda \int \frac{t^3}{2} dt} \right) = \frac{\sin \left( \frac{\lambda (x^3 - 1)/3}{\lambda x} \right)}{\lambda x}. \]

1.2 b. Now we wish to find the second term in the approximation, which will now look like
\[ \left( v_0 \pm \frac{v_1}{\lambda} \right) e^{\pm i \int \frac{p(t)}{x} dt} \]
where we know \( v_1 \) is the following.

\[
v_1 = \pm \frac{i}{2\sqrt{P}} \int^x \frac{1}{\sqrt{P}} \left( \frac{1}{\sqrt{P}} \right)'' \, dt = \pm \frac{i}{2x} \int^x \frac{1}{t} 2t^{-3} \, dt = \pm \frac{i}{x} \cdot \frac{1}{-3x^3} = \mp \frac{i}{3x^4}.
\]

Hence

\[
y_1(x) = A e^{i\lambda} \int^x t^2 \, dt \left( \frac{1}{x} - \frac{i}{3\lambda x^4} \right) + B e^{-i\lambda} \int^x t^2 \, dt \left( \frac{1}{x} + \frac{i}{3\lambda x^4} \right).
\]

And we find \( A \) and \( B \) again:

\[
y_1(1) = A \left( 1 - \frac{i}{3\lambda} \right) + B \left( 1 + \frac{i}{3\lambda} \right) = 0 \Rightarrow B = -A \frac{1 - i/3\lambda}{1 + i/3\lambda}.
\]

\[
y_1'(1) = i\lambda A \left( 1 - i/3\lambda \right) + A \left( -1 + 4i/3\lambda \right) - i\lambda B \left( 1 + i/3\lambda \right) + B \left( -1 - 4i/3\lambda \right) = 1
\]

\[
\Rightarrow 2i\lambda A \left( 1 - i/3\lambda \right) \left( 1 + i/3\lambda \right) + A \left( -1 + 4i/3\lambda \right) \left( 1 + i/3\lambda \right) + A \left( 1 + 4i/3\lambda \right) \left( 1 - i/3\lambda \right) = (1 + i/3\lambda)
\]

\[
\Rightarrow A \left( 2i\lambda + 2i/9\lambda + 6i/3\lambda \right) = (1 + i/3\lambda)
\]

Hence

\[
A = C \left( 1/3\lambda - i \right), \quad B = C \left( 1/3\lambda + i \right), \quad C = \frac{1}{2\lambda 1 + 10/9\lambda^2},
\]

and we obtain the second-order solution:

\[
y_1(x) = 2C \sin \left( \lambda (x^3 - 1)/3 \right) \left( \frac{1}{x} + \frac{1}{9\lambda^2x^4} \right) + \frac{2C}{3\lambda} \cos \left( \lambda (x^3 - 1)/3 \right) \left( \frac{1}{x} - \frac{1}{x^4} \right).
\]

We present below the numerical comparisons. We see that the accuracy of each solution gets better as \( \lambda \) becomes larger, and that the second-order solution is significantly better than the first-order one, but only for larger \( \lambda \).

### 2 Chapter 7, Problem 10

The particle is moving in the potential

\[
V = \frac{1}{2} \kappa x^4
\]

The turning points are obtained by setting

\[
E - V(x) = 0
\]

\[
x^4 = \frac{2E}{\kappa},
\]

which gives \( x_1 = -L, \ x_0 = L \), where \( L = \left( \frac{2E}{\kappa} \right)^{\frac{1}{4}} \).

We have

\[
I = \lambda \int_{-L}^{L} \sqrt{E - \frac{1}{2} \kappa x^4} \, dx
\]
Putting $X = x/L$, we have

$$I = \lambda \int_{-1}^{1} \sqrt{E - \frac{1}{2}\kappa \left(\frac{2E}{\kappa} X^4\right)} LdX$$

$$= \lambda E^{3/4} \left(\frac{2}{\kappa}\right)^{1/4} \int_{-1}^{1} \sqrt{1 - X^4} dX$$

$$= \lambda E^{3/4} \left(\frac{2}{\kappa}\right)^{1/4} \cdot 2 \int_{0}^{1} \sqrt{1 - X^4} dX$$

Consider the integral

$$\int_{0}^{1} \sqrt{1 - X^4} dX$$

Apply the change of variable

$$t = X^4$$

$$dt = 4X^3 dX$$

$$dX = \frac{1}{4} t^{-3/4} dt$$

And we can evaluate the integral by expressing it as a Beta function,

$$\int_{0}^{1} \sqrt{1 - X^4} dX = \frac{1}{4} \int_{0}^{1} t^{-3/4} \sqrt{1 - t} dt$$

$$= \frac{1}{4} \int_{0}^{1} t^{-1/2} (1 - t)^{3/2 - 1} dt$$

$$= \frac{\Gamma(1/4)\Gamma(3/2)}{4\Gamma(7/4)}$$

This gives us

$$I = \lambda E^{3/4} \left(\frac{2}{\kappa}\right)^{1/4} \frac{\Gamma(1/4)\Gamma(3/2)}{2\Gamma(7/4)}$$

By the derivation in the text, we have

$$I = \left(n + \frac{1}{2}\right) \pi, \quad \text{where } n = 0, 1, 2, ...$$

Thus the approximate quantum energy is

$$E_n = \frac{2\kappa^{1/3}}{\lambda^{4/3}} \left[ \frac{\Gamma(7/4) \left(n + \frac{1}{2}\right) \pi}{\Gamma(1/4)\Gamma(3/2)} \right]^{4/3}, \quad \text{where } n = 0, 1, 2, ...$$

(1)