1

(a) We have \((D^2 - x^2)y = 0\). The dimension of \(D^2\) is \(-2\), the dimension of \(x^2\) is \(2\). (b) We write \(y(x) = \sum_n a_n x^n\). Hence \(y'' = \sum_n n(n-1) a_n x^{n-2}\), and so the differential equation gives us

\[
\sum_n n(n-1) a_n x^{n-2} = \sum_n a_n x^{n+2},
\]

or

\[
n(n-1) a_n = a_{n-4}
\]

(1)

Since there are no singular points, \(a_n = 0\) for \(n < 0\). Using \(n = 2\) in (1) we see that \(2a_2 = a_{-2} = 0\), and in fact \(a_{4k+2} = 0\) for any integer \(k\). In the same way, using \(n = 3\) in (1) we see that \(a_{4k+3} = 0\) for any integer \(k\). And using \(n = 0, 1\) in (1) indicates that \(a_0, a_1\) are undetermined. We shall write the solution in terms of these two constants. Consider \(a_{4k}\) for any integer \(k > 0\):

\[
a_{4k} = \frac{a_{4(k-1)}}{4^2 k(k-1/4)} = \ldots = \frac{a_0}{4^2 k!(k-1/4)\ldots(3/4)} = \frac{a_0 \Gamma(3/4)}{4^2 k! \Gamma(k+3/4)}.
\]

Now consider \(a_{4k+1}\) for any integer \(k > 0\):

\[
a_{4k+1} = \frac{a_1}{4^2 k!(k+1/4)\ldots(5/4)} = \frac{a_1 \Gamma(5/4)}{4^2 k! \Gamma(k+5/4)}.
\]

Note that there are only 2 different dimensions in the equation, so that we obtain a recursion formula involving 2 terms \((a_n\) and \(a_{n-4}\)), which can be written down explicitly. And the solution is

\[
y = \sum_{k=0}^{\infty} \frac{a_0 \Gamma(3/4)x^{4k}}{4^2 k! \Gamma(k+3/4)} + \frac{a_1 \Gamma(5/4)x^{4k+1}}{4^2 k! \Gamma(k+5/4)}.
\]

2

(a) We want to find the WKB approximation of \(y\) for \(x > x_0\), and for which values of \(x\) do we expect it to be a good approximation. First, we have

\[
p = \sqrt{x} \Rightarrow \int p(x')dx' = \int_{x_0}^x x p(x')dx' = \frac{2}{3} \left( x^{3/2} - x_0^{3/2} \right).
\]
Important note: the technique works fine if you used \( \int p(x') \, dx' = \frac{2}{3} x^{3/2} \) instead, but finding the constant is much more messy... We expect this to be valid when
\[
\left| \frac{d}{dx} \right| \ll 1 \Rightarrow \left| -\frac{1}{2x^{3/2}} \right| \ll 1 \Rightarrow x \gg 2^{-2/3} \approx 1.
\]

Hence we write
\[
y_{WKB} = A \cos \left( \int p(x') \, dx' \right) + B \sin \left( \int p(x') \, dx' \right) = \frac{A \cos \left( 2x^{3/2}/3 - 2x_0^{3/2}/3 \right) + B \sin \left( 2x^{3/2}/3 - 2x_0^{3/2}/3 \right)}{x^{1/4}}.
\]

And we use the initial conditions to solve for \( A \) and \( B \):
\[
y_{WKB}(x_0) = 1 = \frac{A}{x_0^{1/4}} \Rightarrow A = x_0^{1/4}
\]
\[
y_{WKB}'(x_0) = 0 = \frac{B \cos(0)\sqrt{x}x^{1/4} - A \cos(0)x^{-3/4}/4}{x_0^{1/2}} \Rightarrow B = x_0^{-5/4}/4.
\]

So that
\[
y_{WKB} = \left( \frac{x_0}{x} \right)^{1/4} \cos \left( 2x^{3/2}/3 - 2x_0^{3/2}/3 \right) + \frac{1}{4x^{1/4}x_0^{5/4}} \sin \left( 2x^{3/2}/3 - 2x_0^{3/2}/3 \right).
\]

(b) We now wish to compare the WKB approximation we just computed with the actual solution, which we shall obtain using a numerical solver such as Matlab’s ode45. Careful: we need to be careful for large values of \( x \), because the numerical solution is less and less accurate as \( x \to \infty \). If you would like to see the code I used to plot things, send me an email. See plots below, where we see the error slowly decrease as \( x \) becomes larger.

3 Chapter 7, Problem 4

\[
\frac{d^4 y}{dx^4} + \lambda^4 U(x) y = 0, \quad \lambda \gg 1
\]

First, consider the case \( U(x) > 0 \), let
\[
p(x) = \left[ U(x) \right]^{1/4},
\]
and \( P(x) = \int_0^x p(x') \, dx' \) (note that the lower limit does not matter)

The zeroth-order WKB approximation is
\[
y_0(x) = e^{\alpha P(x)}, \quad \text{for some number } \alpha
\]

Then
\[
y_0(x) = \alpha \lambda p(x) e^{\alpha P(x)} = \alpha \lambda p(x) e^{\alpha \int_0^x p(x') \, dx'}
\]
\[
y_0^{(4)}(x) = \alpha^4 \lambda^4 p^4(x) e^{\alpha \int_0^x p(x') \, dx'} + O(\lambda^3)
\]
both solutions, \( x_0 = 10 \)
Putting this into the differential equation, and compare the $O(\lambda^4)$ terms, we require

$$\alpha^4 + 1 = 0$$

or

$$\alpha = e^{i\pi/4}, e^{3i\pi/4}, e^{5i\pi/4}, e^{7i\pi/4}.$$

Thus,

$$y^{WKB}_0(x) = e^{\frac{1+i\sqrt{2}}{2} \lambda P(x)}, e^{-\frac{1+i\sqrt{2}}{2} \lambda P(x)}, e^{\frac{1-i\sqrt{2}}{2} \lambda P(x)}, e^{-\frac{1-i\sqrt{2}}{2} \lambda P(x)}.$$

To find higher-order terms of the solutions, we put

$$y = e^{\alpha \lambda P(x)} v,$$

into the differential equation, where $\alpha$ is one of the four values determined above. Carrying out the calculations, we have

$$d^4 \left[ e^{\alpha \lambda P(x)} \right] dx^4 = e^{\alpha \lambda P(x)} (D + \lambda \alpha p)^4 v$$

Substituting the expression above into the differential equation, we find that the $O(\lambda^4)$ terms cancel, giving us

$$[4(\alpha \lambda p)^3 D + 6(\alpha \lambda)^3 p^2 p' + O(\lambda^2)] v = 0$$

Dividing by $\lambda^3$ and letting $\epsilon = 1/\lambda$, we get

$$\left[ D + \frac{3p'}{2p} + O(\epsilon) \right] v = 0 \quad (2)$$

We solve the equation above by regular perturbation. Let

$$v = v_0 + \epsilon v_1 + \ldots,$$

and requiring the $O(1)$ terms of equation (4) to vanish, we get

$$\left( D + \frac{3p'}{2p} \right) v_0 = 0$$

$$v_0(x) = [p(x)]^{-3/2}$$

Thus, for $U(x) > 0$, we get the zeroth and first order WKB approximations as:

$$y^{WKB}_0(x) = e^{\alpha \lambda \int^x U(x')^{1/4} dx'} \frac{1}{U(x)^{3/8}}, \quad (3)$$

where $\alpha = e^{i\pi/4}, e^{3i\pi/4}, e^{5i\pi/4}, e^{7i\pi/4}$.

We can carry out similar calculations for the case $U(x) < 0$ to obtain

$$y^{WKB}_0(x) = e^{\beta \lambda \int^x |U(x')|^{1/4} dx'} \frac{1}{|U(x)|^{3/8}}, \quad (4)$$

where $\beta = \pm 1, \pm i$. 