Problem Set 6

Unless otherwise specified, you may use MATLAB to assist with computations. Please provide a print-out of the code used and its output with your assignment.

1. Springs in equilibrium. Consider the following system of springs and masses:

![Diagram of springs and masses]

The black squares are fixed and cannot move. Suppose the masses are \((m_1, m_2, m_3, m_4) = (m, 2m, 3m, 4m)\) and the spring constants are \((c_1, c_2, c_3, c_4, c_5) = (3c, 4c, c, 2c, c)\).

If masses 1 and 4 are each pulled to the right by a force \(f\), what are the resulting displacements of the four masses from the initial equilibrium in terms of \(m\), \(c\), and \(f\)? How much is each spring stretched or compressed? What is the force in each spring?

2. Springs in motion. Again consider the line of springs from Problem (1). This time, let \((m_1, m_2, m_3, m_4) = (m, 2m, 2m, m)\) and \((c_1, c_2, c_3, c_4, c_5) = (c, 2c, c, 2c, c)\).

(a) Set up the differential equation

\[
Mu''(t) + Ku(t) = F(t)
\]

governing the displacements \(u(t)\) of the four masses. Leave the applied forces \(F\) unknown for now.

(b) Assume there is no applied force. Find all of the ways the masses can oscillate with a single fixed frequency. That is, find all solutions to Eq. (1) with \(F = 0\) of the form

\[
u(t) = A \cos(\omega t - \delta) x,
\]

where \(x\) is a vector, \(A\), \(\omega\), and \(\delta\) are scalars, and none of them depend on \(t\). These are called normal modes of the system.

Which quantities are determined by the differential equation? Which will be determined by the initial conditions?

(c) Still with \(F = 0\), find \(u(t)\) if the initial conditions are

\[
u(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u'(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.
\]
3. Second order linear differential equations. Consider the equation

\[ mu''(t) + bu'(t) + cu(t) = f(t) \]  \hspace{1cm} (2)

with \( m, b, \) and \( c \) all positive constants.

(a) Set up a forward Euler scheme for solving this equation numerically. That is, convert Eq. (2) to a first order equation for a vector \( x \) and give an explicit formula for \( x_{n+1} = x(n\Delta t) \) in terms of \( x_n, t_n, m, b, c, \) and \( f. \)

(b) For what values of \( A \) and \( k \) is \( u(t) = Ae^{kt} \) a solution to Eq. (2) with \( f(t) = 0? \) The imaginary parts of these \( k \)'s are the natural frequency of the system.

(c) Now let \( f(t) = \cos(\omega t) \) and \( b = 0. \) Find a solution to Eq. (2) of the form \( u(t) = B \cos(\omega t). \) What happens when \( \omega \) is near the natural frequency you found in the previous part? (This phenomenon is called resonance.)

4. Complex numbers

(a) Write the following complex numbers as complex exponentials \( re^{i\theta}. \)

   i. 1  \hspace{1cm} iii. 1 + i  \hspace{1cm} v. \hspace{0.1cm} -4 - 4i
   ii. i  \hspace{1cm} iv. \hspace{0.1cm} -\sqrt{3} + 3i  \hspace{1cm} vi. \hspace{0.1cm} -4 + 4i

(b) Write the following complex numbers in the form \( a + bi \) with real \( a \) and \( b. \)

   i. \hspace{0.1cm} e^{i\pi}  \hspace{1cm} iii. \hspace{0.1cm} 2e^{i\pi/2}
   ii. \hspace{0.1cm} \sqrt{3}e^{-i\pi/6}  \hspace{1cm} iv. \hspace{0.1cm} 2e^{-i\pi/2}

(c) Write the real (a) and imaginary (b) parts of a complex number \( z = a + bi \) in terms of \( z \) and its complex conjugate \( z^* = a - bi. \) Use this to write \( \sin(\theta) \) and \( \cos(\theta) \) in terms of the complex exponentials \( e^{i\theta} \) and \( e^{-i\theta}. \)

(d) Using complex exponentials, prove the following trigonometric identity:

\[ (\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta). \]

5. Fourier series

Consider function \( g(t) \) which has period \( 2\pi \) and satisfies \( g(t) = t \) for \( t \in (-\pi, \pi). \)

(a) Sketch \( g(t) \) on the interval \((-2\pi, 2\pi).\)

(b) Find the Fourier series of \( g(t). \)

(c) How fast are the Fourier coefficients of \( g(t) \) decaying?