Problem Set 4

Unless otherwise specified, you may use MATLAB to assist with computations. Please provide a print-out of the code used and its output with your assignment.

1. Projections.
   (a) Find the point on the plane $2x + 2y + z = 0$ nearest to the point $(x = 3, y = 3, z = -2)$.
   (b) Point $a$ has coordinates $(x = 0, y = 1, z = 5)$ and point $b$ has coordinates $(x = 4, y = 4, z = 4)$. Which of these points is closer to the line defined by $\{y = 2x, y = 2z\}$?

2. Least squares. This problem is to be done by hand. Consider the system of equations $Au = b$ where $A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$.
   (a) Show that there are no exact solutions for $u$.
   (b) Set up the normal equations for the vector $\hat{u}$ that minimizes the length of $b - A\hat{u}$.
   (c) Find the $QR$ factorization, $A = QR$ using the Gram-Schmidt procedure. Use it to solve for $\hat{u}$.
   (d) What best-fit problem does this least squares problem correspond to? Sketch the data points and the best-fit curve.

3. Linear Transformations
   (a) Consider a Householder matrix of form $H = I - 2uu^\top$ where $u$ is a $n \times 1$ unit vector.
      i. Show that $H$ is symmetric ($H^\top = H$), orthogonal ($H^\top H = I$), and that it is its own inverse ($H = H^{-1}$). Is $H$ full rank?
      ii. Find all eigenvalues of $H$.
   (b) Consider the matrix $D = I - uu^\top$ where $u$ is a $n \times 1$ unit vector.
      i. Describe the transformation of a vector $v$ when multiplied on the left by the matrix $D$. Similarly, describe the transformation $D^{ij}v$.
      ii. Is $D$ full rank? If not, find an orthonormal basis for the null space of $D$.
      iii. Does $D$ have a zero eigenvalue? If so, how many? Find the associated eigenvector.
4. Consider the following “internet” with five websites, where each arrow indicates a link from its starting page to its destination page.

(a) Write the eigenvalue equation $Ar = \lambda r$ that defines the ranks $r_1, \ldots, r_5$ of the five websites. What value of $\lambda$ do we want?

(b) Solve this system to find the page ranks. Which page is trusted most?

5. Population dynamics. **This problem is to be done by hand.** A rabbit population ($r$) and wolf population ($w$) satisfy the following pair of coupled differential equations:

$$\frac{dr}{dt} = 10r - 4w, \quad (1)$$

$$\frac{dw}{dt} = 2r + w. \quad (2)$$

If the initial number of rabbits is 25 and the initial number of wolves is 15, what are the populations at time $t$? After a long time, what is the ratio of rabbits to wolves?