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(1) (5 × 6 = 30 points.)
Consider the truss drawn on the board.

a) Is there a possible mechanism? If so, draw one.
b) Build the matrix A corresponding to this truss.
c) Use the matrix A to verify mathematically your answer to (a). That is, depending on your answer to (a): either prove the nullspace of A is trivial, or give a basis for the nullspace of A (no need to prove you have indeed a basis, but be sure of your answer!).
d) Now fix node 2. Get rid of bar 3 since it is now useless. What is the matrix A for this new truss?
e) Is there a mechanism for the truss in (d)? Prove this mathematically.

\[ A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \]

\[ (\text{rigid motion}) \]

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\[ \begin{pmatrix} u_1^H \, u_1^V \end{pmatrix} = \begin{pmatrix} u_2^H \, u_2^V \end{pmatrix} = -u_2^V \Delta \]

\[ \text{Hence a basis for the nullspace is the given, } \hat{u} \]

\[ \text{(The 3 rows of } A \text{ are independent hence } A \text{ has rank 3 hence the dimension of the nullspace is } 4 - 3 = 1, \text{ so we are done.)} \]

\[ A = \begin{pmatrix}
-\sqrt{2} & \sqrt{2} \\
-1 & 0 \\
\end{pmatrix} \] (upper left block of old A!

\[ (\text{so got rid of last 2 cols, last row}) \]

d) No mechanism! No rigid motion! A is invertible (cols are independent, rows are too, det (A) = +\sqrt{2} \neq 0), so the nullspace is trivial, the truss is stable.
(2) \((3 \times 8 + 2 \times 3 = 30\) points.)

Consider a hanging bar. We have the following equation for its displacement \(u\), given that \(c(x) = 1/2\) for \(x < 1/2\) and \(c(x) = 2\) for \(x > 1/2\), and \(f(x) = \delta(x - \gamma_0)\) (constant force of 1 applied to every part of the bar):

\[
-\frac{d}{dx} \left( c(x) \frac{du}{dx} \right) = f(x), \quad x \in [0, 1].
\]

We also have the boundary conditions \(u(0) = 0\) and \(\frac{du}{dx}(1) = 0\).

a) Find \(w(x) = c(x) \frac{du}{dx}\), and graph it.

\[
w(x) = - \int_0^x \delta(s - \gamma_0) \, ds = \begin{cases} \frac{C_1}{2} & x < \frac{1}{2} \\
\frac{1}{2} - C_1 & \frac{1}{2} < x < \frac{3}{2} \end{cases}
\]

\[
\frac{du}{dx}(1) = 0 \Rightarrow w(1) = 0 = -1 + C_1 \Rightarrow C_1 = 1
\]

b) Find \(\frac{dw}{dx}\), and graph it.

\[
\frac{dw}{dx} = \frac{w(x)}{c(x)} = \begin{cases} 2 & x < \frac{1}{2} \\
0 & \frac{1}{2} \leq x < 1 \\
1 & x \geq 1
\end{cases}
\]

c) Find \(u\), and graph it.

\[
u = \int_0^x \frac{dw}{dx} \, ds = \begin{cases} \int_0^x 2 \, ds & x < \frac{1}{2} \\
\int_0^{1/2} 2 \, ds + \int_0^x 0 \, ds & \frac{1}{2} < x \leq 1
\end{cases}
\]

\[
\text{and } u(0) = 0 = 2 \cdot 0 + C_2 \Rightarrow C_2 = 0 \text{ so } u(x) = \begin{cases} 2x & x < \frac{1}{2} \\
1 & \frac{1}{2} \leq x < x
\end{cases}
\]

Circle your answers to (d), (e) (no explanation).

d) Which part of the bar is stiffer: \(0 < x < 1/2\) or \(1/2 < x < 1\).

e) Which part of the bar is stretched (more) \(0 < x < 1/2\) or \(1/2 < x < 1\).
We know that a system of springs and masses can be modeled using the $K = A^TCA$ framework we have seen in class, in particular, $Mu'' = -Ku$ for $u$ the vector of displacements of the masses. We use primes to mean time derivatives. $A$ is the first difference matrix and $C$ is the constitutive law.

a) What does the equation $Mu'' = -Ku$ for $u$ simplify to if we only have one mass and one spring? (Use the notation on the board.)

$$m u'' = -k u \quad (M, A, C \text{ are } 1 \times 1 \text{ matrices } : M = m, C = k, A = I)$$

b) Find an analytical solution to the equation $mu'' = -ku$, for $u = u(t)$. Don’t bother with initial conditions. Just give ONE expression (there could be more than one that work, it could be complex) for $u$ which satisfies the given equation.

$$u(t) = e^{-i \omega t}, \quad \omega = \pm \sqrt{k/m} \quad \text{(guess } u = e^{i \omega t} \text{ then } u' = i \omega e^{i \omega t},$$

$$u'' = (i \omega)^2 e^{i \omega t} = -\omega^2 e^{i \omega t} \quad \text{so } \quad u'' = -\omega^2 e^{i \omega t} = -|k/m| e^{i \omega t}$$

$$= 0 \quad \omega = \pm \sqrt{k/m} \quad \text{, } u = Ae^{i \omega t}, \quad \sin \omega t = \cos \omega t \quad \text{would work too} \ldots$$

BONUS! → c) Now find a solution of $mu'' = -ku$, for $u = u(t)$, with the following initial conditions: $u(0) = 1$ and $u'(0) = 0$.

We know $\cos 0 = 1$, $\frac{d}{dt}(\cos) \bigg|_0 = -\sin 0 = 0$, so we guess

$$u(t) = \cos \omega t, \quad \omega = \pm \sqrt{k/m}, \quad \text{and indeed it works:}$$

$$u'(t) = -\sin \omega t \quad \omega \quad u'' = -\omega^2 \cos \omega t = -|k/m| \cos \omega t \quad \text{so } \quad \omega = \pm \sqrt{k/m} \quad \text{as \ in \ (b), and } u(0) = \cos 0 = 1, \quad u'(0) = -\omega \sin \omega t \bigg|_{t=0} = -\omega \cdot 0 = 0$$

d) We want to solve the problem in (c) with $m = 1$ (so we will ignore $m$) and $k = 4$ using the leap-frog method (finite differences in time). Modify lines 4, 5 and 9 (don’t modify anything else) of the following Matlab code to do this.

```matlab
01: T=2*pi;nt=50;dt=T/nt;
02: k=4;
03: u=zeros(1,nt+1);
04: u0=1;
05: up0=0;
06: u(1)=u0;
07: u(2)=u0+dt*up0;
08: for i=3:nt+1
09:    u(i)=2*u(i-1) - u(i-2) + (dt^2) * (-k*u(i-1));
end
10: end
```
(4) (16 points.)

(a) I want to use Newton's method to solve \( p(x) = 0 \) with starting guess \( x_0 \). Write down the algorithm to find the next guesses \( x_1, x_2, \) etc. (1 line, plus say for which indices that line holds).

\[
x_{n+1} = x_n - \frac{p(x_n)}{\frac{\partial p}{\partial x}(x_n)}, \quad n = 0, 1, \ldots
\]

(Think \( u_{n+1} = u_n - \frac{\partial q}{\partial u}(u_n), \quad n = 0, 1, \ldots \))

(b) I want to solve Laplace's equation \( -\frac{\partial^2 u}{\partial x^2} = f \) on \( x \in [0, 1] \) with boundary conditions (CAREFUL!) \( u(1) = 0 \) and \( \frac{du}{dx}(0) = 0 \). Let \( h = 1/3 \). Draw the hat functions you would use to solve this problem using the Finite Element Method. Label your axes. OK to draw them all on same graph.

(\( \phi \)'s were flipped from what we had done in class, so flip your \( \phi \)'s!)

All \( \phi \)'s satisfy \( \phi(1) = 0 \)

\[ u(x) = u_1 \phi_1(x) + u_2 \phi_2(x) + u_3 \phi_3(x) \]