Problem Set 3

Unless otherwise specified, you may use MATLAB to assist with computations. Please provide a print-out of the code used and its output with your assignment.

1. Suppose that $Au = b$ has a solution $u$. Show that $\hat{u} = u$ is a solution to the normal equations. What must be true about $A$ for $\hat{u}$ to be unique?

2. Projections.
   (a) Find the point on the plane $x + y + z = 0$ nearest to the point $(x = 1, y = 2, z = 3)$.
   (b) Point $a$ has coordinates $(x = 5, y = 0, z = 1)$ and point $b$ has coordinates $(x = 3, y = 3, z = 3)$. Which of these points is closer to the line defined by $\{x = 2y, y = 2z\}$?

3. Least squares. This problem is to be done by hand. Consider the system of equations $Au = b$ where $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$.
   (a) Show that there are no exact solutions for $u$.
   (b) Setup and find the solution $\hat{u}$ to the normal equations that minimizes the length of the vector $b - A\hat{u}$.
   (c) Find the QR factorization, $A = QR$ using the Gram-Schmidt procedure. Use it to solve for $\hat{u}$.
   (d) What best-fit problem does this least squares problem correspond to? Sketch the data points and the best-fit curve.

4. Linear transformations.
   (a) Consider a Householder matrix of form $H = I - 2uu^T$ where $u$ is a $n \times 1$ unit vector.
      i. Show that $H$ is symmetric ($H^T = H$), orthogonal ($H^TH = I$), and that it is its own inverse ($H = H^{-1}$). Is $H$ full rank?
      ii. Find all eigenvalues of $H$.
   (b) Consider the matrix $D = I - uu^T$ where $u$ is a $n \times 1$ unit vector.
      i. Describe the transformation of a vector $v$ when multiplied on the left by the matrix $D$. Similarly, describe the transformation $D^2v$.
      ii. Is $D$ full rank? If not, find an orthonormal basis for the null space of $D$.
      iii. Does $D$ have a zero eigenvalue? If so, how many? Find the associated eigenvector.
5. Population dynamics. **This problem is to be done by hand.** A rabbit population $(r)$ and wolf population $(w)$ satisfy the following pair of coupled differential equations:

\[
\frac{dr}{dt} = 6r - 2w, \quad (1)
\]

\[
\frac{dw}{dt} = 2r + w. \quad (2)
\]

If the initial number of rabbits is 30 and the initial number of wolves is 30, what are the populations at time $t$? After a long time, what is the ratio of rabbits to wolves?