1. $A^T A$ is at least positive semidefinite.
   - If $A$ has independent columns, $A^T A$ is positive definite.
   - This means that $A x = 0$ has only the solution $x = 0$ (zero vector).
   - Reasoning: Use the energy test for positive semidefiniteness.
   - $x^T A^T A x \geq 0$ why?
   - $(A x)^T (A x) = \text{length squared of vector } A x$.
   - Automatically $\geq 0$.
   - Zero energy when $A x = 0$. If $A$ has independent columns then $x = 0$. Energy test passed.

2. Four big points of linear algebra
   A. Solve $K u = f$ by elimination.
   B. From independent vectors $v_1, \ldots, v_r$ compute orthonormal vectors $b_1, \ldots, b_r$.
   C. $K x = A v$ - The eigenvectors diagonalize $K$.
   D. $A v = \Sigma u$ - The singular vectors $v$ and $u$ diagonalize any $A$.
      Singular vectors $v$'s and $u$'s diagonalize any $A$.
      We can choose $v$'s = eigenvectors of $A A^T$ orthonormal
      $u$'s = eigenvectors of $A^T A$ orthonormal
      $A = U \Sigma V^T$
If \( A \) was symmetric, then \( v^s = u^s = \text{eigenvectors of } A^t A \).

If \( A \) is rectangular we must go for \( A^t A \) instead. The same eigenvalues \( \lambda \geq 0 \).

Section 18.8: Applications of the SVD

This is not an official part of 18.085! Optional!

Applications to data matrices (rectangular)

samples \( x \) to

\[
A = \begin{bmatrix}
82 & 71 & 93 & 77 & 42 & 100 \\
\end{bmatrix}
\]

columns = students, each row gives each row gives

undergraduate grades in 1 course

Principal Component Analysis = \( \text{PCA} \)

Physics grades on row \( 1 \), all students in \( 1 \) course

smart choice

So that the

\[ y \text{ times } A = \text{most important part of } A \]

\[ A \times A = \text{most information} \]

\[ \text{compressed into just 2 vectors } v \text{ and } u \]