Problem Set 4
Due: October 29, 2015 (E18-366, 1:00pm)

Name (Print):

Instructions:

• Include a printed copy of this document with your solutions.
• Organize your work, in a neat and coherent way. Work without a clear ordering will receive very little credit.
• A correct answer, unsupported by calculations, explanation, or algebraic work will receive very little credit; an incorrect answer supported by substantially correct calculations and explanations will still receive partial credit.
• Box your final answer to each problem.
• All the problems are worth the same amount (20 points/problem).

Problem 1 (A.L.): (Question 5 in section 2.4) For every connected graph, explain why the only solutions to $Au = 0$ are the constant vectors $u = (C, \ldots, C)$.

Problem 2 (A.L.): (Question 6 in section 2.4) The sum down the main diagonal of a matrix is the trace. For the complete graph (with four nodes) and the tree (with four nodes), the traces of $A^T A$ are $3 + 3 + 3 + 3 = 12$ and $1 + 2 + 2 + 1 = 6$. Why is the trace of $A^T A$ always $2m$ for any graph with $m$ edges?

Problem 3 (T.F.):
(1) ([Section 2.7 Problem 4]) Truss D has how many rows and columns in the matrix $A$? Find column 1, with 8 elongations from a small displacement $u_1^H$. Draw a nonzero solution to $Au = 0$. Why will $A^T w = 0$ have a nonzero solution (8 bar forces in balance)?
(2) Recall that the rank of a matrix $A$ is the maximal number of columns of $A$ which are linearly independent. Deduce that the matrix $A$ in (1) has rank 7 by showing that the matrix $A'$ obtained from deleting row 7 and column 6 is invertible. What is the truss represented by $A'$? You may use MATLAB to carry out the matrix calculation, you may also work with other equivalent definitions of rank you can find.
(3) Does truss D in (1) have any mechanism?

Problem 4 (T.F.): (cf. Section 2.7 Problem 7, 10)
We will study space truss in this problem.
(1) Suppose a space truss has $m$ bars, $n_1$ fixed nodes and $n_2$ free nodes. What is the size of
matrix $A$?
(2) What is the number of rigid motions in 3D? Can you describe 3D rigid motions?
(3) If a space truss is stable, what is its minimal number of fixed nodes? Why?
(4) Consider a tetrahedron truss (6 bars) with 4 nodes located at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ respectively. Suppose that all the nodes lying on the $z = 0$ plane are fixed. Write down the matrix $A$. Is this truss stable?

Problem 5 (S.C.):
(1) Solve $u'' = -x$ with $u(0) = u(1) = 0$. then solve approximately with two hat functions and $h = \frac{1}{3}$. Where is the biggest error? Do they agree on the nodes?
(2) Let $\phi$ be the parabola of height 1 with zeros at $3h$ and $4h$. Use Simpson’s rule to compute the area under $\phi$ and the area under $(\phi')^2$.
(3) Let $(\phi_i)$ be a set of independent trial functions such that $\phi_i(0) = \phi_i(1) = 0$. Check that the matrix $K = (\int_0^1 \phi_i \phi_j' \, dx)_{i,j}$ is positive definite.

Problem 6 (S.C.):
If $\Phi$ a finite set of independent functions, we will consider $E(\Phi) = \int_0^1 (u' - u_\Phi')^2$ as a measure of the error in the approximation.

Let $(P)$ be the fixed-fixed problem $u'' + 2u' + u = x + 2$ with $u(0) = u(1) = 0$

(1) What is the exact solution of $(P)$? What is its weak formulation? How is the approximation $u_\Phi$ defined given a set of independent test functions $\Phi$? (Find a matrix $K$ and a vector $F$ such that $u_\Phi = \sum U_i \phi_i$ where $U = (U_i)$ is the solution of $KU = F$)

We will start with linear approximations

(2) Let $n$ be an integer and $\Phi^{(n)} = \{\phi_1, \ldots, \phi_{n-1}\}$ be the set of functions defined as follows: $\phi_i(x) = 0$ if $x \notin \left[\frac{i-1}{n}, \frac{i+1}{n}\right]$, $\phi_i\left(\frac{i}{n}\right) = 1$ and $\phi_i$ is linear on both $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ and $\left[\frac{i}{n}, \frac{i+1}{n}\right]$. Sketch what this set of functions looks like. Why are they independent? Compute $K$ and $F$ in this case.

(3) Use MATLAB to compute $U$ when $n = 6$. Plot $u$ with $u_{\Phi^{(6)}}$.

(4) Explain why any piecewise linear functions on the subsegments $\left[\frac{i}{n}, \frac{i+1}{n}\right]$ lies in $H_{\Phi^{(n)}}$ and deduce the order (as a power of $1/n$) of $E(\Phi^{(n)})$.  

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