Problem 1.

1) \( A = LU \) is invertible, therefore both \( L \) and \( U \) are as well. 
   \[ A = L^T U \equiv U^T L^T. \] Hence \((U^T)^{-1}L = L^T U^{-1}\) is both upper and lower triangular, i.e. diagonal. Let \( D = U (L^T)^{-1} \).
   Then \( U = D L^T \) as we wanted, and \( A = LDL^T \).

2) In 6.3 terms in G. Strang's book, \( A = LDL^T \) means that all the coefficients of \( D \) are positive.
   \[ \forall X \in \mathbb{R}^n \quad X^T LDL^T X > 0 \Rightarrow X^T DX > 0 \quad \forall X \in \mathbb{R}^n \]
   \[ (L \text{ is invertible}) \]
   \[ \iff \sum d_i x_i^2 > 0 \quad \forall X = (x_i) \in \mathbb{R}^n \]
   \[ \iff d_i > 0 \quad \forall i = 1 \ldots n. \]

Problem 2

\[ \Pi(a) = \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix}, \quad a \in \mathbb{R}. \]

\[ \Pi(1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \text{ are not invertible because its columns are linearly dependent.} \]

\[ \Pi(-1) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \]

* To get \( \Pi(0) \)'s first pivot, we need to switch row 1 and row 3.
   \[ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \Pi(0) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = LDLD^T \]

* To get \( \Pi(-1) \)'s second pivot, we need to switch row 2 and row 3.
   \[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Pi(-1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & -1 \end{pmatrix} = LDLD^T \]
for $a \in \mathbb{Z} \setminus \{0, d, -d^2\}$, $\Pi(a)$ admits a LU-factorization:

\[
\Pi(a) = \begin{pmatrix}
\frac{1}{a} & 0 & 0 \\
\frac{1}{a} & 1 & 0 \\
\frac{1}{a} & \frac{1}{a+1} & 1
\end{pmatrix}
\begin{pmatrix}
a & 0 & 0 \\
0 & \frac{a-1}{a} & 0 \\
0 & 0 & \frac{(a+2)(a+1)}{(a+1)}
\end{pmatrix}
\begin{pmatrix}
1 & \frac{1}{a} & \frac{1}{a} \\
0 & 1 & \frac{1}{a+1} \\
0 & 0 & 1
\end{pmatrix}
\]

$L(a)$ $U(a)$
Problem 3

(1) The general solutions to \(- \frac{du}{dx} = g(x-a)\) are of the form: \(u(x) = -R(x-a) + Cx + D\).

Impose the boundary conditions:

\[ u(0) = 0 \implies D = 0. \]

\[ u(1) = 0 \implies C = 1 \]

So the solution is:

\[ u(x) = -R(x-a) + x \]

(2) The graph of \(u(x)\) is:

The graph of \(u'(x)\) is (the "..." part).
(3). To solve the discrete fixed-free problem, the matrix $H$ to use is obtained from $K$ by changing its lower right corner entry to 1.

i.e. $H_2 = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$, $H_3 = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}$, 

if the distance between two adjacent nodes is $h = \frac{1}{n+1}$, the linear problem we are solving is

$$\frac{1}{h^2} \cdot H_n \cdot u = \frac{S}{h}.$$ 

The "$h$" below $S$ is crucial.

Matlab solution:

$n = 2 \quad u = \begin{pmatrix} 0.3333 \\ 0.3333 \end{pmatrix}$

$n = 3 \quad u = \begin{pmatrix} 0.2500 \\ 0.2500 \\ 0.2500 \end{pmatrix}$

$n = 4 \quad u = \begin{pmatrix} 0.2000 \\ 0.4000 \\ 0.4000 \\ 0.2000 \end{pmatrix}$

$n = 5 \quad u = \begin{pmatrix} 0.1667 \\ 0.3333 \\ 0.3333 \\ 0.3333 \\ 0.1667 \end{pmatrix}$

The plot is as follows.
Problem 4.

(1). \[ S'(f) = \int_{-\infty}^{\infty} S'(x) f(x) \, dx = \int_{-\infty}^{\infty} (S(x) f(x))' \, dx \]

formal integration by part

\[ = \int_{-\infty}^{\infty} S(x) f'(x) \, dx \]

\[ = S(x) f(x) \bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} S(x) f'(x) \, dx \]

f vanishing at \( \infty \).

= \[ \int_{-\infty}^{\infty} S(x) f'(x) \, dx \]

by definition of \( S \)

\[ - f'(c) \]

(2). The argument is almost identical to (1).

By our definition:

\[ S'(f) = \lim_{n \to \infty} \int_{-\infty}^{\infty} f(x) p_n(x) \, dx \]

integration by part

\[ = \lim_{n \to \infty} \left( \int_{-\infty}^{\infty} (f(x) p_n(x))' \, dx - \int_{-\infty}^{\infty} f'(x) p_n(x) \, dx \right) \]

"genuine"
\[
\lim_{n \to \infty} \left( f \circ p_n(x) \right)_{-\infty}^{+\infty} = - \int_{-\infty}^{+\infty} f(x) p_n(x) \, dx
\]

vanishing at \( \infty \)
\[
= - \lim_{n \to \infty} \int_{-\infty}^{+\infty} f(x) p_n(x) \, dx
\]

\( p_n \) approximate \( \delta \)
\[
= - \delta(f')
\]
\[
= \left[ - f'(0) \right]
\]

This definition has the advantage that the "integration by part" is genuine.

(3) We use MATLAB to check that
\[
\int_{-\infty}^{+\infty} \sin(\cos(x)) p_n(x) \, dx \quad \to \quad n \to \infty \quad \sin(1) = 0.84147098 \ldots
\]

The plot of \( \int_{-\infty}^{+\infty} \sin(\cos(x)) p_n(x) \, dx \) against \( n \) is below, where the horizontal line is \( y = \sin(1) \).
Problem 5

A and $A^T$ have the same eigenvalues because

\[
\lambda \text{ is an eigenvalue of } A \text{ if and only if } \det(A - \lambda I) = 0
\]

Further

For any matrix $M$, $\det(M) = \det(M^T)$

therefore

\[
\det(A - \lambda I) \Rightarrow \det(A^T - \lambda I) = 0
\]

But

\[
(A - \lambda I)^T = A^T - \lambda I
\]

hence,

\[
\det(A^T - \lambda I) = 0
\]

whenever

\[
\det(A - \lambda I) = 0
\]
Problem 5 Continued...

If \( A \) is a Markov matrix, each row of \( A^T \) adds to 1. This implies that:

\[
A^T \begin{bmatrix}
1 \\
\vdots \\
1
\end{bmatrix} = \begin{bmatrix}
\sum_{j=1}^{n} (A^T)_{ij} \\
\vdots \\
\sum_{j=1}^{n} (A^T)_{nj}
\end{bmatrix}
\]

but \( \sum_{j=1}^{n} (A^T)_{ij} \) is precisely the \( i^{th} \) row sum of \( A^T \), which is 1, therefore

\[
A^T \begin{bmatrix}
1 \\
\vdots \\
1
\end{bmatrix} = \begin{bmatrix}
1 \\
\vdots \\
1
\end{bmatrix}
\]

so 1 is an eigenvalue of \( A^T \) because

\[
\begin{bmatrix}
1 \\
\vdots \\
1
\end{bmatrix}
\]

is its eigenvector.
Problem 6

(1) If $\sum_{i=1}^{n} \mu_i = 1$, and $\mu_i \geq 0$, we see

$$(Q\mu)_i = \sum_{j=1}^{n} Q_{ij} \mu_j$$

Each $(Q\mu)_i \geq 0$, further, summing over $i$ gives:

$$\sum_{i=1}^{n} (Q\mu)_i = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} Q_{ij} \right) \mu_j$$

$$= \sum_{i=1}^{n} \mu_i = 1$$

where we have used

$$\sum_{i=1}^{n} Q_{ij} = 1$$

$$\sum_{i=1}^{n} Q_{ij} = 1$$
18.085 Solution Set

Pset 2

(2) First, note \( Q = \mathbb{I}_n - C_{n/3} \) where \( C_n \) is the circulant matrix.

Next, we can compute the eigenvalues of \( Q \) from those of \( C_n \) (in section 1.5) as follows

\[
\begin{align*}
\det (\lambda \mathbb{I} - Q) &= 0 \\
\Rightarrow \quad \det (\lambda \mathbb{I} - \mathbb{I} + C_{n/3}) &= 0 \\
\Rightarrow \quad \det ((\lambda - 1) \mathbb{I} + C_{n/3}) &= 0 \\
\Rightarrow \quad (-\frac{1}{3})^n \det (3(1-\lambda) \mathbb{I} - C_n) &= 0
\end{align*}
\]

Now this means: \( \lambda \) is an eigenvalue of \( Q \) if and only if \( 3(1-\lambda) \) is an eigenvalue of \( C_n \). We know the eigenvalues of \( C_n \) (denoted by \( \eta_k \)):

\[
\eta_k = 2 - 2 \cos \left( \frac{2\pi k}{n} \right) \quad k = 0, 1, \ldots, n-1
\]
Thus the eigenvalues of $Q$ (called $\lambda_k$)
satisfy:

$$3(1 - \lambda_k) = 2 - 2\cos\left(\frac{2\pi k}{n}\right) = \gamma_k$$

$k = 0, 1, ..., n-1$

Rearranging, gives

$$\lambda_k = \frac{1}{3} \left(1 + 2\cos\left(\frac{2\pi k}{n}\right)\right)$$

$k = 0, 1, ..., n-1$

Now, we know

$$Q^t = \sum_{i=1}^n \lambda_i^+ v_i v_i^T$$

where

$v_1, ..., v_n$ are an orthonormal set of vectors (also the eigenvectors associated with $\lambda_i$).

Next, we see that all $\lambda_i$ except $\lambda_i = 1$ go to $0$, but remember $Q^T \hat{\mu} = \sum_{i=1}^n \lambda_i^+ v_i v_i^T \hat{\mu}$ but $\hat{\mu} = \sum_{j=1}^n a_j v_j$, therefore all $\lambda_i$, if $i \neq 1$ are less than 1.
\[
Q^T \mu = \sum_{i=1}^{n} a_i \lambda_i^+ \nu_i \\
\rightarrow a_i \nu_i
\]

but by Problem 5, we know

\[\nu_i = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \text{ further by part }\]

(1) we know \[\sum_{j=1}^{n} (a_i \nu_i)_j = 1\]

\[\therefore Q^T \mu \rightarrow \frac{1}{n} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \text{ as claimed.}\]
% Dimension of the matrix Q
n = 100;
% Length of time we run the chain T
T = 100;
% Create the transition matrix
Q = (1/3)*toeplitz([1,1,zeros(1,n-3),1]);
% Store the values of the L^2 norm from t=1, ..., T here
v = zeros(n,1);
% Store the time vector here
t = [1:T];
% Store the value of the L^2 norm of the jth iteration of the chain
% in the j-th component of v
for j=1:T
v(j) = sqrt(sum(((Q^j)*[1,zeros(1,n-1)])' - (1/n)*ones(n,1)).^2));
end
% graph it.
plot(t,v);