Problem Set 1

Problem 1

a) 
\[
\begin{bmatrix}
1 \\
3 \\
2
\end{bmatrix}
\begin{bmatrix}
1 & 3 & 2 \\
3 & 9 & 6 \\
2 & 6 & 4
\end{bmatrix} =
\begin{bmatrix}
1 & 3 & 2 \\
3 & 9 & 6 \\
2 & 6 & 4
\end{bmatrix}.
\]  
(1)

As for why $K$ is singular, one could argue that it has a non-trivial nullspace, which we see in part b).

b) $Ku = 0$ if and only if $Au = 0$, and $A = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}$ is already triangular, so we immediately see that 

\[
Ku = 0 \iff u \in \text{span}\left( \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right).
\]  
(2)

Note that when you are asked to compute a nullspace, it would be nice to provide a basis—in some psets, there were 4 linearly dependent vectors given that indeed span the nullspace, but two would have been sufficient to characterize it.

c) 
\[
K^4 = (A^t A)^4 = A'(AA')^3 A = 14^3 K =
\begin{bmatrix}
2744 & 8232 & 5488 \\
8232 & 24696 & 16464 \\
5488 & 16464 & 10976
\end{bmatrix}.
\]  
(3)

Note that this simple idea—computing matrix products in a numerically opportune order—is also at the heart of the adjoint method.

Problem 2

a) For cubes:

\[
(n + 1)^3 - 2n^3 + (n - 1)^3 = n^3 + 3n^2 + 3n + 1 - 2n^3 + n^3 - 3n^2 + 3n - 1 = 6n, \quad n \in \mathbb{N},
\]  
(4)

which is the correct second derivative, if we interpret the input as polynomial over $\mathbb{R}$, for example.

For quartics:

\[
(n + 1)^4 - 2n^4 + (n - 1)^4 = n^4 + 4n^3 + 6n^2 + 4n + 1 - 2n^4 + n^4 - 4n^3 + 6n^2 - 4n + 1 = 12n^2 + 2, \quad n \in \mathbb{N},
\]  
(5)

which is off by 2 from the correct derivative.

b) Squares:

\[
\frac{1}{2} \left( (n + 1)^2 - (n - 1)^2 \right) = \frac{1}{2} \left( n^2 + 2n + 1 - n^2 + 2n - 1 \right) = 2n,
\]  
(6)

\[
\frac{d}{dx} x^2 \bigg|_{x=n} = 2n, \quad n \in \mathbb{N}.
\]  
(7)
Cubics:

\[(n + 1)^3 - n^3 = n^3 + 3n^2 + 3n + 1 - n^3 = 3n^2 + 3n + 1,\]  
\[\frac{1}{2} \left( (n + 1)^3 - (n - 1)^3 \right) = \frac{1}{2} \left( n^3 + 3n^2 + 3n + 1 - n^3 + 3n^2 - 3n + 1 \right) = 3n^2 + 1,\]  
\[\left. \frac{d}{dx} x^3 \right|_{x=n} = 3n^2,\quad n \in \mathbb{N}.\]

Quartics:

\[(n + 1)^4 - n^4 = n^4 + 4n^3 + 6n^2 + 4n + 1 - n^4 = 4n^3 + 6n^2 + 4n + 1,\]  
\[\frac{1}{2} \left( (n + 1)^4 - (n - 1)^4 \right) = \frac{1}{2} \left( n^4 + 4n^3 + 6n^2 + 4n + 1 - n^4 + 4n^3 - 6n^2 + 4n - 1 \right) = 4n^3 + 4n,\]  
\[\left. \frac{d}{dx} x^4 \right|_{x=n} = 4n^3,\quad n \in \mathbb{N}.\]

Problem 3

Use the ansatz from class, \(u(x) = -R(x-a) + Ax + B\) and fit to the boundary conditions: 0 = \(u(0) = B\), 0 = \(u'(1) = A - 1\), so \(A = 1\). Combined, we get

\[u(x) = \begin{cases} x, & x \in [0,a), \\ a, & x \in [a,1], \end{cases}\]

which means \(u\) first grows linearly and at \(a\) becomes constant.

We can plot it for e.g. \(a = 0.4\) with

```matlab
1 a = 0.4;
2 x = linspace(0,1,100);
3 plot(x, (x < a).*x + (x >= a).*a);
```

Figure 1: Plot of \(u\) for \(a = 0.4\)
Problem 4

For example, apply centered second differences twice to get

\[
\frac{1}{(h^2)^2} ( (u_{i+2} - 2u_{i+1} + u_i) - 2(u_{i+1} - 2u_i + u_{i-1}) + (u_i - 2u_{i-1} + u_{i-2}) )
\]

\[
= \frac{1}{h^4} (u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}),
\]

the desired coefficients. For the analysis of the corresponding matrix, we sometimes consider it without the
\(h^{-4}\) factor, but when actually solving a problem, we have to keep it there.

Problem 5

We can solve it with Matlab:

```matlab
n = 5;
h = 1/n;
e = ones(n-1,1);

% Generate K and D (we choose centered first differences) as sparse
% matrices
K = spdiags([-e 2*e -e], -1:1, n-1, n-1);
D = spdiags([-e zeros(n-1,1) e], -1:1, n-1, n-1)/2;

% Right-hand side is just ones
b = ones(n-1,1);
u = (1/(h^2) * K + 1/h * D);
u = [0; u; 0];
```

This yields

\[
u = \begin{bmatrix}
0 \\
0.0714 \\
0.1141 \\
0.1220 \\
0.0871 \\
0
\end{bmatrix}.
\]  

Problem 6

By Gaussian elimination (or using Matlab), we see

\[
T_3 = \begin{bmatrix}
3 & 2 & 1 \\
2 & 2 & 1 \\
1 & 1 & 1
\end{bmatrix}, \quad T_4 = \begin{bmatrix}
4 & 3 & 2 & 1 \\
3 & 3 & 2 & 1 \\
2 & 2 & 2 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix},
\]

whence we guess the general form

\[
(T_n^{-1})_{ij} = \begin{cases}
  n - i + 1, & i \geq j, \\
  n - j + 1, & j > i,
\end{cases} \quad 1 \leq i, j \leq n.
\]

In order to verify that this is indeed the inverse, look at what happens when we multiply by \(T_n\) and compute \((T_nT_n^{-1})_{ij}\): It is enough to look at one row of \(T_n\), and we note that there are only three cases for the corresponding entries in \(T_n^{-1}\), namely \((k, k + 1, k + 2)', (k, k + 1, k + 1)'\) and \((k, k, k)'\) for some integer \(k\), for the
interior rows \( 1 < i < n \). Checking the result yields:

\[
\begin{bmatrix}
-1 & 2 & -1 \\
\end{bmatrix}
\begin{bmatrix}
k \\
k+1 \\
k+2 \\
\end{bmatrix}
= -k + 2(k + 1) - (k + 2) = 0
\]  

(20)

\[
\begin{bmatrix}
-1 & 2 & -1 \\
\end{bmatrix}
\begin{bmatrix}
k \\
k+1 \\
k+1 \\
\end{bmatrix}
= -k + 2(k + 1) - (k + 1) = 1
\]  

(21)

\[
\begin{bmatrix}
-1 & 2 & -1 \\
\end{bmatrix}
\begin{bmatrix}
k \\
k \\
k \\
\end{bmatrix}
= -k + 2k - k = 0.
\]  

(22)

Since the case \((k, k + 1, k + 1)^\prime\) corresponds to the diagonal element, we have the correct result for the identity matrix.

The calculations for the first and the last row, \(i = 1\) or \(i = n\), respectively, are similar.