YOUR NAME: ________________________________

YOUR SCORE: __________ / 100 + __________ / 20 extra credit

THE QUIZ IS OPEN BOOK, OPEN NOTES AND NO CALCULATORS

GRADING:

(1) 1. __________

(2) 2. __________ + __________ BONUS POINTS

(3) 3. __________

(4) 4. __________
The dft of the signal $x$ is
\[ \hat{x} = \begin{pmatrix} 3 \\ 0 \\ 2i \\ 0 \\ 0 \\ 0 \\ 0 \\ 2i \\ 0 \end{pmatrix} \]

What signal is sampled in $x$? Sketch the real and the imaginary part of $x$. (With our usual notation, here we have that $N=11$).

**Solution:**

As
\[ x = G_N \hat{x}, \]

it follows that
\[ x = \sum_{k=0}^{N-1} \hat{x}_k v_k = 3v_0 + 2iv_2 + 2iv_9 \]
\[ = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + 2i \begin{pmatrix} 1 \\ e^{1\cdot2\pi i/11} \\ e^{2\cdot2\pi i/11} \\ \vdots \\ e^{10\cdot2\pi i/11} \end{pmatrix} + 2i \begin{pmatrix} 1 \\ e^{-1\cdot2\pi i/11} \\ e^{-2\cdot2\pi i/11} \\ \vdots \\ e^{-10\cdot2\pi i/11} \end{pmatrix} = \begin{pmatrix} 3 + 4i \\ 3 + 4i \cos\left(\frac{2\cdot2\pi}{11}\right) \\ 3 + 4i \cos\left(\frac{4\cdot2\pi}{11}\right) \\ \vdots \\ 3 + 4i \cos\left(\frac{20\cdot2\pi}{11}\right) \end{pmatrix}. \]

The graph of the real part is the graph of a $f(t) = 3$. The graph of the imaginary part is the graph of $g(t) = 4 \cos(2t)$ and the points we are sampling are those corresponding to $t = 0, \frac{2\pi}{11}, \frac{2\cdot2\pi}{11}, \ldots, \frac{10\cdot2\pi}{11}$. 
(2) (15 + 15 + 10 points +10 + 10 bonus points.)
Which of these equations can be solved? If you can solve them, then show a solution, otherwise explain why it is not possible to find a solution.

a) \[
\operatorname{div} \left( \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \right) = y^2 - 3\lambda x - 1,
\]
where \( \lambda \) is a fixed real parameter.
b) The Laplace equation on the unit circle, with boundary condition,

\[-\Delta u = 0, \quad u(1, \theta) = \cos(2\theta) + \sin(6\theta) + 1.\]
c) \[
\operatorname{div} \left( -\frac{\partial s}{\partial y} \frac{\partial s}{\partial x} \right) = e^{x^2-3y},
\]

**Solution:**

a) A solution to

\[\operatorname{div} \left( \frac{\partial u}{\partial x} \right) = \Delta u = y^2 - 3\lambda x - 1\]

is given by \( u(x, y) = \frac{y^4}{12} - \frac{\lambda x^3}{2} - \frac{x^2}{2} \).
b) A solution to

\[-\Delta u = 0, \quad u(1, \theta) = \cos(2\theta) + \sin(6\theta) + 1.
\]
is given by

\[u(r, \theta) = r^2 \cos(2\theta) + r^6 \sin(6\theta) + 1\]
as shown in the book at page .
c) The equation

\[\operatorname{div} \left( -\frac{\partial s}{\partial y} \frac{\partial s}{\partial x} \right) = e^{x^2-3y},\]

has no solution, since

\[\text{div} \left( -\frac{\partial s}{\partial y} \right) = -\frac{\partial^2 s}{\partial x \partial y} + \frac{\partial^2 s}{\partial y \partial x} = 0.\]
Caveat: the next two are bonus questions.

d1) Find a family of curves $s(x, y) = C$ that is everywhere orthogonal to the family of curves $u(x, y) = e^{-y}(\sin(x))$.

d2) Can you tell for which complex function the function $u(x, y)$ in d1) is the real part?

Solution:

d1) As $\text{div}(\nabla u) = -e^{-y} \sin(x) + e^{-y} \sin(x) = 0$, then $u$ admits a stream function $s$, which is determined by the two conditions

$$
\frac{\partial s}{\partial y} = \frac{\partial u}{\partial x} = e^{-y} \cos(x),
$$

and

$$
-\frac{\partial s}{\partial x} = \frac{\partial u}{\partial y} = -e^{-y} \sin(x).
$$

Then,

$$
s = \int \frac{\partial s}{\partial y} \, dy + F(x) = \int -e^{-y} \cos(x) \, dy + F(x) = -e^{-y} \cos(x) + F(x),
$$

and

$$
s = \int \frac{\partial s}{\partial x} \, dx + G(y) = \int e^{-y} \sin(x) + G(y) = -e^{-y} \cos(x) + G(y).
$$

Hence $s(x, y) = -e^{-y} \cos(x)$ is a solution.

d2) The complex function we are looking for has $u$ as real part and $s$ as imaginary part, hence

$$
h(x, y) = u(x, y) + is(x, y) = e^{-y} \sin(x) + i(-e^{-y} \cos(x)) = e^{-y}(\sin(x) - i \cos(x)) = -ie^{-y}(\cos(x) + i \sin(x)) = -ie^{iz},
$$

where $z = x + iy$. 

(3) (15 points). Let \( f(x) = -|x| + g(x) \) on \([-\pi, \pi]\), where \( g(x) \) is defined as follows,

\[
g(x) = \begin{cases} 
12 & \text{for } x \in [0, \pi] \\
-2 & \text{for } x \in (-\pi, 0).
\end{cases}
\]

Find the Fourier series of \( f(x) \), when \( f \) is extended periodically to the real line, by the rule \( f(2\pi + x) = f(x) \).

**Solution:**

We can rewrite \( g \) as \( g(x) = 5 + 7SW(x) \), where

\[
SW(x) = \begin{cases} 
1 & \text{for } x \in [0, \pi] \\
-1 & \text{for } x \in (-\pi, 0).
\end{cases}
\]

The Fourier series of \( SW(x) \), that we saw in class multiple times, is

\[
\text{4} \pi \sum_{N \ni k, k \text{odd}} \frac{\sin(kx)}{k},
\]

hence the Fourier series of \( g(x) \) is

\[
g(x) = 5 + \frac{28}{\pi} \sum_{N \ni k, k \text{odd}} \frac{\sin(kx)}{k}.
\]

\( h(x) = -|x| \) is an even function, hence its Fourier series will be a series of cosines, \( \sum_{k=0}^{\infty} c_k \cos(kx) \). Let us compute it.

\[
c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} -|x| \, dx = -\frac{\pi}{2}
\]

\[
c_k = \frac{1}{\pi} \int_{-\pi}^{\pi} -|x| \cos(kx) \, dx = \begin{cases} 
0 & \text{for } k \text{ even} \\
\frac{1}{\pi k^2} & \text{for } k \text{ odd}.
\end{cases}
\]

Hence the Fourier series of \( f(x) \) is

\[
f(x) = -\frac{\pi}{2} + 5 + \sum_{N \ni k, k \text{odd}} \frac{28 \sin(kx)}{\pi k} + \frac{4 \cos(kx)}{\pi k^2}.
\]
(4) (5 + 5 + 5 + 5 + 5 points)

a) Consider the function \( f(x) = e^{-x}, \; x \in [-\pi, \pi] \) which is extended periodically to the real line, by the rule \( f(2\pi + x) = f(x) \). Draw the graph in \([-2\pi, 2\pi]\).

The functions satisfies the following differential equation

\[
\frac{d}{dx}f(x) + f(x) = g(x),
\]

for some function \( g(x) : [-\pi, \pi] \rightarrow \mathbb{R} \). Find such \( g \).

b) Compute the complex Fourier series of \( f(x) \),

\[
f(x) = \sum_{k \in \mathbb{Z}} d_ke^{ikx},
\]

in the standard way.

**Solution:**

![Figure 1. The graph of f.](image)

a) For the differential equation, we have that

\[
\frac{d}{dx}e^{-x} + e^{-x} = -e^{-x} + e^{-x} = 0,
\]

hence \( g(x) \) is the constantly 0 function.
b) 

\[ d_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-x} e^{-ikx} \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-(1+ik)x} \, dx = -\frac{1}{2\pi} \frac{e^{-(1+ik)x}}{(1 + ik)} \bigg|_{-\pi}^{\pi} = \]

\[ = \frac{1}{2\pi} \left[ e^{(1+ik)\pi} - \frac{e^{-(1+ik)\pi}}{(1 + ik)} \right] = \frac{\sinh((1 + ik)\pi)}{\pi(1 + ik)}. \]
c) Try to compute the Fourier coefficients of $f$ using the differential equation (0.1). Do you get the same result as in part b?
d) Explain the reason for the answer you gave in part c.
e) Compute
\[ \sum_{k \in \mathbb{Z}} |d_k|^2. \]

Solution:
c) Applying the differential equations to the Fourier coefficients yields the following equations
\[ d_k(ik) + d_k = 0, \text{ for all } k \in \mathbb{Z}, \ k \neq 0. \]
These equations imply that for $|k| \neq 1$, then $d_k = 0$, which is different from what we obtained in part b).
d) The problem is that $f$ is not differentiable when we look at it as a function on the real line. Hence, as the differential equations (0.1) holds only in $(-\pi, \pi)$ and the discontinuity at $-\pi$ is the source of the discrepancy in the results between part b) (the right one) and part c) (the wrong one).
e) \[
\sum_{k \in \mathbb{Z}} |d_k|^2 = \int_{-\pi}^{\pi} |f(x)|^2\,dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-2x}\,dx = \frac{e^{2\pi} - e^{-2\pi}}{4\pi}.\]