Deteminants. Cross Product.

February 5

Reading Material: From Simmons: 18.3. From Course Notes D.

Last time: Vectors. Vector arithmetic. Dot product.
Today: Cross product. Determinant.

2 Cross Product

Yesterday in your recitation you learned that the dot product of two vectors can be also expressed using the coordinates of the vectors, that is if \( \vec{A} = (a_1, a_2, a_3), \vec{B} = (b_1, b_2, b_3) \), are two vectors in 3D forming an angle \( \theta \), then

\[
\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos \theta = a_1 b_1 + a_2 b_2 + a_3 b_3.
\]

Observe that there is an equivalent formula in 2D.

We now introduce a different kind of product of vectors. This one can only be defined for 3D vectors.

Definition 1. The Cross Product of two vectors \( \vec{A} \) and \( \vec{B} \) (only for 3D) is defined as

\[
\vec{A} \times \vec{B} = (|\vec{A}||\vec{B}| \sin \theta) \hat{n}
\]

where \( \hat{n} \) is the unit vector \( \perp \) to both \( \vec{A} \) and \( \vec{B} \) that satisfies the Right Hand Rule (RHR)\(^1\)

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\(^1\)Think about a screw!
Handy Facts

1. $|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}||\sin \theta| = \text{area of parallelogram spanned by the vectors } \vec{A} \text{ and } \vec{B}$.

2. $\vec{A} \times \vec{B}$ by construction gives a vector orthogonal both to $\vec{A}$ and $\vec{B}$. This will be very important in the future!

To compute $\vec{A} \times \vec{B}$ with coordinates we need a new mathematical tool: determinant.

3 Determinants

Definition 2. A $m \times n$ (m x n) matrix $A$ is a table of scalars

$$
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & \cdots \\
  \vdots & \ddots \\
  a_{m1} & \cdots & a_{mn}
\end{pmatrix}
$$

This matrix has $n$ columns and $m$ rows. If $n = m$ we say that the matrix is square. If the matrix is square ($m = n$) then it has a magic # called determinant. We denote this number by

$$\det A \text{ or } |A|.$$

In the 1 x 1 case $A = (a)$ for some scalar $a$ and we simply have

$$\det A = a.$$

In the 2 x 2 case

$$A = \begin{pmatrix}
  a_1 & a_2 \\
  b_1 & b_2
\end{pmatrix}$$

we define $\det A$ as

$$\det A = a_1b_2 - a_2b_1.$$

Exercise 1.

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} =$$

In the 3 x 3 case

$$A = \begin{pmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3
\end{pmatrix}$$

then

$$\det A = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2$$

Exercise 2.

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 7 & 1 \\ 1 & 9 & 4 \end{vmatrix} =$$
Handy Facts

To calculate determinants the following facts are quite useful:

1. Exchanging 2 rows $\rightarrow$ det flips sign
2. 2 (or more) identical rows in the matrix $\rightarrow$ det $= 0$
3. add/subtract a row from another $\rightarrow$ no change in det

Same considerations for columns.

**Remark.** You will see in recitation that determinants are useful in order to calculate volumes. In fact you will see that if

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

$\vec{A} = (a_1, a_2, a_3), \vec{B} = (b_1, b_2, b_3) \text{ and } \vec{C} = (c_1, c_2, c_3)$, then

$$|\det A| = \text{Volume of parallelogram spanned by } \vec{A}, \vec{B} \text{ and } \vec{C}.$$

Another calculation method: cofactor expansion method

Here I am going to compute the determinant of a $3 \times 3$ matrix by a cofactor expansion relative to the first row:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_{11}m_{11} - a_{12}m_{12} + a_{13}m_{13} = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}.$$

where

$m_{ij} = i, j \text{ minor } = \text{det after remove row } i \text{ col } j$

$c_{ij} = i, j \text{ cofactor } = (-1)^{i+j}m_{ij}$
Once the minors with respect to a certain row have been found then to compute the cofactors one just needs to remember the following distribution of signs on a $3 \times 3$ matrix:

\[
\begin{pmatrix}
  + & - & + \\
  - & + & - \\
  + & - & + 
\end{pmatrix}
\]

**Theorem 1. Computing $\vec{A} \times \vec{B}$**

If $\vec{A} = (a_1, a_2, a_3)$ and $\vec{B} = (b_1, b_2, b_3)$ then

\[
\vec{A} \times \vec{B} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
 a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3
\end{vmatrix}.
\]

**Exercise 3.** Given two vectors $\vec{A} = (3, -2, 4)$ and $\vec{B} = (2, 1, -2)$ find a vector $\vec{N} \perp$ to both $\vec{A}$ and $\vec{B}$. 


Study Guide 1. The questions for this lecture are:

• What is the geometric meaning of the cross product?

• Is the determinant defined for rectangular matrices?

• How do you compute the volume of a parallelepiped spanned by three vectors?

• Two vectors are said to be parallel if one is a scalar (non zero) multiple of the other one. Are the two vectors in Ex. 3 parallel? If I had given two parallel vectors in Ex. 3, for example \( \vec{A} = (0, -2, 4) \) and \( \vec{C} = (0, -1, 2) \), what would have happened? Think about this question both in a geometric way (using a picture) and in an analytic way by computing the determinant.