PROBLEM SET 7 (DUE IN LECTURE ON OCT 29 (THURSDAY))

(All Theorem and Exercise numbers are references to the textbook by Apostol; for instance “Exercise 1.15-3” means Exercise 3 in section 1.15.)

Problem 1. Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) and differentiable on \((a, b)\). Prove that there exists \( c \in (a, b) \) such that

\[
f'(c) + c = \frac{f(b) - f(a)}{b - a} + \frac{a + b}{2}.
\]

(Hint: apply the Mean Value Theorem to an appropriate function.)

Problem 2. Find the maximum and minimum values of the function

\[ f(x) = 2\sqrt{x} + \sqrt{1-x} \]
on the interval \([0, 1]\).

Problem 3. Find the maximum value of the function

\[ f(x) = x + \frac{1}{x^2} + \frac{2}{x + 1} \]
on the interval \([1, 2]\). (Hint: is \( f \) convex?)

Problem 4. Define \( f : \mathbb{R} \to \mathbb{R} \) by the integral

\[ f(x) = \int_{x}^{x^2} \frac{1}{1 + t^2} dt. \]

Compute the derivative \( f'(x) \). (It will be a rational function of \( x \).)

Problem 5. Compute the integral

\[ \int_0^1 \frac{x^3}{\sqrt{1 + x^2}} dx. \]

(Hint: guess a function \( f(x) \) such that \( f'(x) = \frac{x^3}{\sqrt{1 + x^2}} \).)

Problem 6. Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous and thrice-differentiable (so \( f', f'', f''' \) all exist everywhere in \( \mathbb{R} \)). Let \( c \in \mathbb{R} \) be a point such that \( f'(c) = f''(c) = 0 \) and \( f'''(c) > 0 \).

(a) Prove that there exists \( \delta > 0 \) such that \( f''(x) > 0 \) for \( c < x < c + \delta \) and \( f''(x) < 0 \) for \( c - \delta < x < c \).

(b) Prove that \( f'(x) > 0 \) for \( 0 < |x - c| < \delta \), where \( \delta \) is as in part (a).

(c) Conclude that \( f \) does not have a local maximum or minimum at \( c \).