Problem 1 (1.15-3). Show that \( \int_a^b [x] \, dx + \int_a^b [-x] \, dx = a - b \).

Solution. We rewrite the left side as \( \int_a^b ([x] + [-x]) \, dx \), then examine the integrand.

Lemma. By the notation defined in Problem 4, \([x] + [-x] = \chi_Z(x) - 1\).

Proof. We consider the two cases suggested by the characteristic function separately. If \(x \in \mathbb{Z}\), then \([x] = x\) and \(-x \in \mathbb{Z}\), so \([-x] = -x\), hence \([x] + [-x] = x + (-x) = 0\). Meanwhile, \(\chi_Z(x) - 1 = 1 - 1 = 0\) so the formula holds for \(x \in \mathbb{Z}\).

If \(x \not\in \mathbb{Z}\), then let \([x] = y \in \mathbb{Z}\), which satisfies \(y \leq x < y + 1\). Since \(x \not\in \mathbb{Z}\), \(x \neq y\) and \(y < x < y + 1\). Multiplying by \(-1\) and rearranging, \(-y - 1 < -x < -y = (-y - 1) + 1\). Therefore, by definition, \([-x] = -y - 1\) and \([x] + [-x] = y + (-y - 1) = -1\). Meanwhile, \(\chi_Z(x) - 1 = 0 - 1 = -1\), so the formula holds for \(x \not\in \mathbb{Z}\). \(\square\)

Now \(\chi_Z(x) - 1\) is a step function with steps at each integer. On the open intervals of the form \((m, m + 1)\) for \(m \in \mathbb{Z}\) (as well as \((a, m)\) and \((m, b)\) at the ends), it takes on the value of \(-1\). Since the sum of the widths of these intervals is \(b - a\), we have by definition

\[ \int_a^b (\chi_Z(x) - 1) \, dx = (-1)(b - a) = a - b, \]

as desired. \(\square\)

Problem 2 (1.15-5a). Prove that \(\int_0^2 [t^2] \, dt = 5 - \sqrt{2} - \sqrt{3}\).

Solution. We just need to write \(f(t) = [t^2]\) as a step function on \([0, 2]\). Indeed, squaring gives

\[
\begin{align*}
0 \leq t < 1 & \implies 0 \leq t^2 < 1 \\
1 \leq t < \sqrt{2} & \implies 1 \leq t^2 < 2 \\
\sqrt{2} \leq t < \sqrt{3} & \implies 2 \leq t^2 < 3 \\
\sqrt{3} \leq t < 2 & \implies 3 \leq t^2 < 4.
\end{align*}
\]

Therefore, we can write \(f(t)\) as a step function:

\[
f(t) = \begin{cases} 
0 & \text{if } 0 \leq t < 1 \\
1 & \text{if } 1 \leq t < \sqrt{2} \\
2 & \text{if } \sqrt{2} \leq t < \sqrt{3} \\
3 & \text{if } \sqrt{3} \leq t < 2 \\
4 & \text{if } t = 2.
\end{cases}
\]
By the definition of the integral of a step function, therefore,
\[
\int_0^2 f(t) \, dt = 0(1 - 0) + 1(\sqrt{2} - 1) + 2(\sqrt{3} - \sqrt{2}) + 3(2 - \sqrt{3})
\]
\[
= \sqrt{2} - 1 + 2\sqrt{3} - 2\sqrt{2} + 6 - 3\sqrt{3}
\]
\[
= 5 - \sqrt{2} - \sqrt{3},
\]
as desired.

\[ \blacksquare \]

**Problem 3** (1.26-20). Compute the following integral:

\[
\int_{-2}^{-4} (x + 4)^{10} \, dx.
\]

[Hint: Theorem 1.18.]

**Solution.** As the hint suggests, we translate the integral by 4 using Theorem 1.18.

\[
\int_{-2}^{-4} (x + 4)^{10} \, dx = \int_0^2 x^{10} \, dx.
\]

To write this in the form we know, we need to switch the bounds of integration using Theorem 1.19. Then we can simply apply Theorem 1.15:

\[
\int_0^2 x^{10} \, dx = -\int_0^2 x^{10} \, dx = -\frac{2^{11}}{10+1} = -\frac{2048}{11}.
\]

\[ \blacksquare \]

**Problem 4.** Compute the integral

\[
\int_{-1}^{1} \frac{1+x+x^2}{1+x^2} \, dx.
\]

**Solution.** Noticing the symmetric integral, we cleverly split the integral into its even and odd components:

\[
\int_{-1}^{1} \frac{1+x+x^2}{1+x^2} \, dx = \int_{-1}^{1} \left( \frac{1+x^2}{1+x^2} + \frac{x}{1+x^2} \right) \, dx = \int_{-1}^{1} 1 \, dx + \int_{-1}^{1} \frac{x}{1+x^2} \, dx.
\]

The first integrand is constant, so has value 1(1 - (−1)) = 2. To handle the second integral, we will use the following lemma:

**Lemma.** If \( f(x) \) is odd \((f(−x) = −f(x))\) and integrable on \([-a,a]\), then \(\int_{-a}^{a} f(x) \, dx = 0\).

**Proof.** Indeed, we can rewrite using the reflection property:

\[
\int_{-a}^{a} f(x) \, dx = \int_{-a}^{a} f(a+(-a)-x) \, dx = \int_{-a}^{a} f(-x) \, dx = -\int_{-a}^{a} f(x) \, dx
\]

if \( f \) is odd. Therefore, \( 2\int_{-a}^{a} f(x) \, dx = 0 \) so \( \int_{-a}^{a} f(x) \, dx = 0 \) as desired.

\[ \blacksquare \]

To apply this lemma, we need to demonstrate that \( f(x) = \frac{x}{1+x^2} \) is odd and integrable on \([-1,1]\). Indeed, \( \frac{-x}{1+(-x)^2} = -\frac{x}{1+x^2} \), so \( f(−x) = −f(x) \) as desired. Moreover, on \([-1,1]\), it is increasing: If \(-1 \leq x \leq y \leq 1\), \( |xy| = |x| \cdot |y| \leq 1 \) so \( xy \leq 1 \) and therefore

\[
0 \leq (y-x)(1-xy)
\]
\[
0 \leq y-x-xy^2+x^2y
\]
\[
x+xy \leq y+x^2y
\]
\[
x(1+y^2) \leq y(1+x^2)
\]
\[
\frac{x}{1+x^2} \leq \frac{y}{1+y^2}
\]
where the last inequality requires $1 + y^2, 1 + x^2 \geq 1 > 0$. Since our function is increasing, it is integrable. Therefore by the lemma, $\int_{-1}^{1} \frac{x}{1 + x^2} \, dx = 0$ and we are left with

$$\int_{-1}^{1} \frac{1 + x + x^2}{1 + x^2} \, dx = 2 + 0 = 2.$$ 

\[ \square \]

**Problem 5.** The characteristic function (also called the indicator function) of a subset $A \subseteq \mathbb{R}$ is the function $\chi_A : \mathbb{R} \to \mathbb{R}$ defined by

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Prove that $\chi_Q$ is not integrable on the unit interval $[0, 1]$. (You may use the results of Exercises I 3.12-6 and I 3.12-9 without proof.)

**Solution.** We will prove that $\chi_Q$ is not integrable on $[0, 1]$ by showing the upper and lower integrals are not equal.

**Lemma.** $\overline{T}(\chi_Q) = 1$.

**Proof.** First, $\chi_Q(x) \leq 1$ for all $x \in [0, 1]$, so the constant (step) function 1 is an upper bound. Its integral is $\int_0^1 1 \, dx = 1$, so $\overline{T}(\chi_Q) \leq 1$.

Now let $t(x) \geq \chi_Q(x)$ be a step function, and let $[a, b] \subseteq [0, 1]$ be one of the steps. Then $t(x) = c$ is constant on $(a, b)$. By Exercise I 3.12-6, there is some rational number $r \in (a, b)$, so $c = t(r) \geq \chi_Q(r) = 1$. Therefore, $t(x) = c \geq 1$ on this step, and hence on $[0, 1]$. Therefore by the Comparison Theorem, $\int_0^1 t(x) \, dx \geq \int_0^1 1 \, dx = 1$.

Since this holds for any upper bound step function $t(x)$, $\overline{T}(\chi_Q) \geq 1$. With both bounds, $\overline{T}(\chi_Q) = 1$.

**Lemma.** $\underline{I}(\chi_Q) = 0$.

**Proof.** This proof is very similar to the previous one. First, $\chi_Q(x) \geq 0$ for all $x \in [0, 1]$, so the constant (step) function 0 is a lower bound. Its integral is $\int_0^1 0 \, dx = 0$, so $\underline{I}(\chi_Q) \geq 0$.

Now let $s(x) \leq \chi_Q(x)$ be a step function, and let $[a, b] \subseteq [0, 1]$ be one of the steps. Then $s(x) = c$ is constant on $(a, b)$. By Exercise I 3.12-9, there is some irrational number $r \in (a, b)$, so $c = t(r) \leq \chi_Q(r) = 1$. Therefore, $s(x) = c \leq 0$ on this step, and hence on $[0, 1]$. Therefore by the Comparison Theorem, $\int_0^1 s(x) \, dx \leq \int_0^1 0 \, dx = 0$.

Since this holds for any lower bound step function $s(x)$, $\underline{I}(\chi_Q) \leq 0$. With both bounds, $\underline{I}(\chi_Q) = 0$.

Having shown that $\overline{T}(\chi_Q) = 1 \neq 0 = \underline{I}(\chi_Q)$, we conclude from the definition of integrability that $\chi_Q$ is not integrable on $[0, 1]$.

**Problem 6.** Let $S_0 = [0, 1]$ be the closed unit interval and let $S_{i+1}$ be defined by removing the open middle third of each interval in $S_i$. Thus $S_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $S_2 = [0, \frac{1}{3}] \cup [\frac{2}{9}, \frac{2}{3}] \cup [\frac{4}{9}, \frac{5}{9}] \cup [\frac{7}{9}, 1]$, and so on; in general $S_i$ is the union of $2^i$ closed intervals of length $3^{-i}$. The Cantor set $C$ is defined as the intersection of all the $S_i$:

$$C := \bigcap \{ S_i \mid i \geq 0 \}.$$ 

Prove that $\int_0^1 \chi_C(x) \, dx = 0$ where $\chi_C$ is the characteristic function (as defined in the previous problem) of the Cantor set.

**Solution.** We must demonstrate that $\overline{T}(\chi_C) \leq 0$ and $\underline{I}(\chi_C) \geq 0$. Since $\overline{T}(\chi_C) \geq \underline{I}(\chi_C)$, this will be enough to show that both are 0 and hence $\int_0^1 \chi_C(x) \, dx = 0$.

The second bound is easier. Since $\chi_C(x) \geq 0$, we can take the constant (step) function lower bound $s(x) = 0$. Since $\int_0^1 s(x) \, dx = 0$, $\underline{I}(\chi_C) \geq 0$. 

3
For the upper bound, we construct a sequence of upper bound step functions $t_i(x) \geq \chi_C(x)$ with $\int_0^1 t_i(x) \, dx$ decreasing to 0. We can take these step functions directly from the definition of $C$: Let $t_i(x) = \chi_{S_i}(x)$. Since $S_i$ is a disjoint union of $2^i$ intervals, $\chi_{S_i}(x)$ is a step function with steps consisting of the $2^i$ intervals composing $S_i$ and the $2^i - 1$ “deleted middle third” intervals between pairs of $S_i$.

Since $C = \bigcap \{S_i : i \geq 0\}$, $C \subset S_i$ for all $i$. This means that $\chi_C(x) \leq \chi_{S_i}(x)$: If $x \in C$, $x \in S_i$ so $\chi_C(x) = 1 = \chi_{S_i}(x)$ and if $x \notin C$, $\chi_C(x) = 0 \leq \chi_{S_i}(x)$. Therefore, $t_i(x)$ is indeed an upper bound step function for $\chi_C(x)$.

Finally, we compute $\int_0^1 t_i(x) \, dx$. Since $S_i$ consists of $2^i$ closed intervals of width $3^{-i}$, we can write $S_i = \bigcup_{k=1}^{2^i} [a_i, a_i + 3^{-i}]$. Then we can write

$$\int_0^1 t_i(x) \, dx = \sum_{k=1}^{2^i} \int_{a_i}^{a_i + 3^{-i}} \chi_{S_i}(x) \, dx + \sum_{k=1}^{2^i-1} \int_{a_i + 3^{-i}}^{a_{i+1}} \chi_{S_i}(x) \, dx$$

$$= \sum_{k=1}^{2^i} \int_{a_i}^{a_i + 3^{-i}} 1 \, dx + \sum_{k=1}^{2^i-1} \int_{a_i + 3^{-i}}^{a_{i+1}} 0 \, dx$$

$$= \sum_{k=1}^{2^i} (3^{-i}) + \sum_{k=1}^{2^i-1} 0 = 2^i \cdot 3^{-i} = \left(\frac{2}{3}\right)^i.$$

Therefore $\bar{T}(\chi_C) \leq \left(\frac{2}{3}\right)^i$ for all $i \in \mathbb{Z}^+$. Finally, we need to show that the infimum of these numbers is 0:

**Lemma.** $\inf \left\{ \left(\frac{2}{3}\right)^i : i \geq 0 \right\} = 0$.

**Proof.** As the product of positive numbers, $\left(\frac{2}{3}\right)^i \geq 0$, so 0 is a lower bound and therefore the infimum $y \geq 0$ exists. Then $\frac{3}{2}y$ is also a lower bound: Otherwise there is some $i$ such that $\left(\frac{2}{3}\right)^i > \frac{3}{2}y$, but then $\left(\frac{2}{3}\right)^{i+1} > y$, making $y$ not a lower bound. But if $y > 0$, then $\frac{3}{2}y > y$ so $y$ cannot be the smallest lower bound. Hence we must have the infimum $y = 0$ as desired. \(\square\)

Therefore, $\bar{T}(\chi_C) \leq \inf \left\{ \left(\frac{2}{3}\right)^i : i \geq 0 \right\} = 0$, so as we reasoned above, $\int_0^1 \chi_C(x) = 0$ as desired. \(\square\)