MATH 18.01, SPRING 2013 - PROBLEM SET # 7

Professor: Terrence Blackman

Due: by Wednesday on 4-17-13
(Please submit to recitation instructor at beginning of recitation on 4-17-13)

18.01 Supplementary Notes (including Exercises and Solutions) are available on the course
web page: [http://math.mit.edu/classes/18.01/](http://math.mit.edu/classes/18.01/). This is where to find the exercises labeled 1A, 1B, etc. You will need these to do the homework.

**Part I** consists of exercises given and solved in the Supplementary Notes. You are required to write
up, in your own words, solutions to these exercises.

**Part II** consists of problems for which solutions are not given; it is worth more points. Some of
these problems are longer multi-part exercises. They are given as a means to more fully explore ideas
presented in the course. Items and ideas will appear in these exercises that have been not explicitly
discussed in the Lectures. It is my/our expectation that you will do the required independent reading
to give meaningful responses to these problems. Please see the guidelines below for acceptable modes
of collaboration.

You are encouraged to use MAPLE to aid your search for solutions to these problems
and to explore calculus.

**Part I** (20 points)

**Notation:** The problems come from three sources: the Supplementary Notes, the Simmons book,
and problems that are described in full detail inside of this pset. I refer to the former two sources
using abbreviations such as the following ones: 2.1 = Section 2.1 of the Simmons textbook; Notes G =
Section G of the Supplementary Notes; Notes 1A: 1a, 2 = Exercises 1a and 2 in the Exercise Section
1A of the Supplementary Notes; Section 2.4: 13 = Problem 13 in Section 2.4 of Simmons, etc.

Second Fundamental Theorem. New definition of \( \ln(x) \).

*Read:* Notes PI, p.2 [eqn.(7) and example]; Notes FT.

*Homework:* 3E: 1, 3a; Notes 3D: 1, 4bc, 5, 8a; Notes 3E: 2ac.

Areas between curves. Volumes by slicing.

*Read:* 7.1, 7.2, 7.3.

*Homework:* Notes 4A: 1b, 2, 4; Notes 4B: 1de, 6, 7.

Volumes by disks and shells.

*Read:* 7.4.

*Homework:* Notes 4B: 2eg, 5; Notes 4C: 1, 2, 3; Notes 4J: 3.
Part II (50 points)

Directions and Rules: Collaboration on problem sets is encouraged, but:

i) Attempt each part of each problem yourself. Read each portion of the problem before asking for help. If you don’t understand what is being asked, ask for help interpreting the problem and then make an honest attempt to solve it.

ii) Write up each problem independently. On both Part I and II exercises you are expected to write the answer in your own words. You must show your work; “bare” solutions will receive very little credit.

iii) Write on your problem set whom you consulted and the sources you used. If you fail to do so, you may be charged with plagiarism and subject to serious penalties.

iv) It is illegal to consult materials from previous semesters.

0. (not until due date; 3 points) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say none or no consultation. This includes visits outside recitation to your recitation instructor. If you don’t know a name, you must nevertheless identify the person, as in, “tutor in Room 2-106,” or “the student next to me in recitation.” Optional: note which of these people or resources, if any, were particularly helpful to you.

This “Problem 0” will be assigned with every problem set. Its purpose is to make sure that you acknowledge (to yourself as well as others) what kind of help you require and to encourage you to pay attention to how you learn best (with a tutor, in a group, alone). It will help us by letting us know what resources you use.

1. (Second Fundamental Theorem; $3 + 2 + 2 + 3 + 3 + 3 = 16$ points) Let $\text{sinc}(x)$ denote the “sinc” function

$$\text{sinc}(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{\sin x}{x} & \text{if } x \neq 0. \end{cases}$$

Now consider the “sine integral” function

$$\text{Si}(x) = \int_0^x \text{sinc}(t) \, dt.$$ 

Both of these functions frequently come up in Fourier analysis and signal processing and hence have been given their own names. Remark: $\text{Si}(x)$ cannot be expressed in terms of standard elementary functions.

a) Compute $\text{Si}'(x)$ and $\text{Si}''(x)$. You will have to compute $\text{Si}''(0)$ by using the definition of the derivative. Hint: In computing $\text{Si}''(0)$, you can make use of the fact that $\sin(\Delta x) = \Delta x + O((\Delta x)^3)$.

b) List the critical points of $\text{Si}(x)$ in the entire range $-\infty < x < \infty$. Which critical points are local maxima and which ones are local minima?

c) Draw a rough sketch of $\text{Si}'(x)$ and $\text{Si}''(x)$. The drawings only have to be qualitatively correct, but make sure that the zeros of $\text{Si}'(x)$ are accurately displayed.
d) Sketch the graph of $\text{Si}(x)$ on the interval $-10\pi \leq x \leq 10\pi$ with labels for the critical points and inflection points. The drawing should be qualitatively correct and should reflect the shape of the graphs you sketched in part c).

e) Let $r > 1$ be a real number, and define

$$f(x) = \begin{cases} 
0 & \text{if } x = 0, \\
\frac{\sin(x^r)}{x} & \text{if } x \neq 0.
\end{cases}$$

Remark: It is not too hard to show that $f(x)$ is continuous, even at $x = 0$. Consider the function

$$h(x) = \int_0^x f(t) \, dt.$$  

Show that $h(x)$ can be expressed in terms of composition of $\text{Si}$ with another function.

f) Compute

$$\lim_{x \to 3} \frac{x^2}{x - 3} \int_3^x \text{sinc}(t) \, dt.$$ 

2. (Volumes by slicing; 5 points) 7.3: 22

3. (Volumes by slicing; 10 points) Find the volume of the three-dimensional solid with $x > 0, y > 0, z > 0$ and

$$z^4 < x + y < z.$$ 

Hint: First find the area of the horizontal cross sections.

4. (Shell and disk method; 2 + 1 + 2 + 1 = 6 points) (Funky donut)

a) Use the cylindrical shell method to express the volume of a donut with square cross sections in terms of an integral. Your answer should depend on $\ell$ and $R$. Here, $R$ is the distance from the center of the (round) hole in the donut to the center of a square cross-section of the donut (where the cream filling is located), and $\ell$ is the side length of the cross sectional squares.

Remark: The hole of the donut is round. It is the cross sections of the donut itself that are square.

b) Then compute the integral to find a formula for the volume.

c) Repeat parts a) and b), but this time using the disk method.

5. (Shell method; 3 + 7 = 10 points) 7.4: 12, 13 Remark: Think of the “spherical ring” as a sphere that has been gored by a cylinder whose radius is smaller than the radius of the sphere, but whose length is infinite.

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