18.01 Supplementary Notes (including Exercises and Solutions) are available on the course web page: [http://math.mit.edu/classes/18.01/](http://math.mit.edu/classes/18.01/). This is where to find the exercises labeled 1A, 1B, etc. You will need these to do the homework.

**Part I** consists of exercises given and solved in the Supplementary Notes. You are required to write up, in your own words, solutions to these exercises.

**Part II** consists of problems for which solutions are not given; it is worth more points. Some of these problems are longer multi-part exercises. They are given as a means to more fully explore ideas presented in the course. Items and ideas will appear in these exercises that have been not explicitly discussed in the Lectures. It is my/our expectation that you will do the required independent reading to give meaningful responses to these problems. Please see the guidelines below for acceptable modes of collaboration.

**You are encouraged to use MAPLE to aid your search for solutions to these problems and to explore calculus.**

**Part I (15 points)**

**Notation:** The problems come from three sources: the Supplementary Notes, the Simmons book, and problems that are described in full detail inside of this pset. I refer to the former two sources using abbreviations such as the following ones: 2.1 = Section 2.1 of the Simmons textbook; Notes G = Section G of the Supplementary Notes; Notes 1A: 1a, 2 = Exercises 1a and 2 in the Exercise Section 1A of the Supplementary Notes; Section 2.4: 13 = Problem 13 in Section 2.4 of Simmons, etc.

Linear and quadratic approximations.

*Read:* Notes A.

*Homework:* Notes 2A: 2, 3, 7, 11, 12ade.

Curve-sketching.

*Read:* 4.1, 4.2.

*Homework:* Notes 2B: 1, 2ae, 4, 6ab, 7ab.
Part II (40 points)

Directions and Rules: Collaboration on problem sets is encouraged, but:

i) Attempt each part of each problem yourself. Read each portion of the problem before asking for help. If you don’t understand what is being asked, ask for help interpreting the problem and then make an honest attempt to solve it.

ii) Write up each problem independently. On both Part I and II exercises you are expected to write the answer in your own words. You must show your work; “bare” solutions will receive very little credit.

iii) Write on your problem set whom you consulted and the sources you used. If you fail to do so, you may be charged with plagiarism and subject to serious penalties.

iv) It is illegal to consult materials from previous semesters.

0. (not until due date; 3 points) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say none or no consultation. This includes visits outside recitation to your recitation instructor. If you don’t know a name, you must nevertheless identify the person, as in, “tutor in Room 2-106,” or “the student next to me in recitation.” Optional: note which of these people or resources, if any, were particularly helpful to you.

This Problem 0 will be assigned with every problem set. Its purpose is to make sure that you acknowledge (to yourself as well as others) what kind of help you require and to encourage you to pay attention to how you learn best (with a tutor, in a group, alone). It will help us by letting us know what resources you use.

1. (quadratic approximations; 3 + 2 + 3 = 8 points) Your electric guitar speaker has a continuous dial that varies from 0 to 11. Assume that the volume output \( v \) of the speaker is a twice differentiable function of the dial number \( d \). Using your decibel meter, you have measured the volume output of your speaker at the following three settings:

\[
\begin{array}{c|c|c|c}
  \text{d} & \text{v (in decibels)} \\
  \hline
  8 & 79.931 \\
  9 & 90.744 \\
  10 & 101.983 \\
\end{array}
\]

You are very curious to know the volume output when \( d = 11 \), but since you live in a dorm, you don’t dare turn the dial to 11. Luckily, you are enrolled in 18.01 and are therefore able to estimate the output using the following procedure.

a) Estimate \( v'(9) \) and \( v'(10) \) by using difference quotients of the form \( \frac{f(x)-f(x-\Delta x)}{\Delta x} \) for \( \Delta x = 1 \).

b) Use part a) and a similar procedure to estimate \( v''(10) \).

c) Derive a quadratic approximation of \( v(d) \) near \( d = 10 \). In your quadratic approximation, you can use the approximate values for \( v' \) and \( v'' \) from parts a) and b). Then use your quadratic approximation to estimate \( v(11) \).

2. (limits; continuity; quadratic approximations; \( 2 + 2 + 1 + 2 + 1 + 1 + 1 = 10 \) points) In this problem, you will investigate what happens to the surface area of a sphere when you slightly squash it. A perfect unit sphere can be visualized in the following way: consider the set of points in the \((x, y)\) plane that satisfy the equation \( x^2 + y^2 = 1 \). This curve describes a sphere of radius 1 centered at the origin. Now imagine that this circle is continuously rotated about the \( y \) axis until it has spun a full
360 degrees. The resulting 3 – d object that is traced out is a sphere of radius 1 in three-dimensional space centered at the origin. Now instead of the circle, consider the curve

\[ x^2 + \left( \frac{y}{a} \right)^2 = 1, \]

where \( a \) is some positive constant with \( 0 < a < 1 \). This curve describes an ellipse in the \((x, y)\) plane centered at the origin. The horizontal axis of the ellipse has length 2 and the vertical axis has length \( 2a \). Now imagine that this ellipse is rotated about the \( y \) axis in the manner described above. The resulting 3 – d object that gets traced out is called an oblate ellipsoid, which means that it is a sphere that has been squashed in one direction (in this case the \( y \) axis is the squashed direction). This geometric object comes up when one studies the shape of the earth: the earth’s rotation gives it a slightly squashed character (in reality the earth is quite bumpy, and therefore the oblate ellipsoid model of the earth may not be accurate enough for all applications).

There is an important quantity associated to the oblate ellipsoid called the eccentricity \( E \). It is 0 for a perfect sphere, and for an oblate ellipsoid, it measures how squashed it is:

\[ E = \sqrt{1 - a^2}. \]

In vector calculus, you will develop the tools necessary to calculate the surface area of the oblate ellipsoid. Here, I will only provide you with the formula. The surface area \( S \) can be expressed in terms of \( E \) as follows:

\[ S = 2\pi \left( 1 + f(E) \times (1 - E^2) \right), \]

where

\[ f(E) = \begin{cases} \frac{\tanh^{-1}(E)}{E} & \text{if } E \neq 0, \\ 1 & \text{if } E = 0. \end{cases} \]

In the above formula, \( \tanh^{-1}(E) \) is the inverse of the hyperbolic tangent function. That is, \( y = \tanh^{-1}(E) \) if and only if \( \tanh(y) = E \), where \( \tanh(E) = \sinh(E)/\cosh(E) \).

a) Show that \( f(E) \) is continuous at \( E = 0 \).

b) Show that \( (d/dE) \tanh^{-1}(E) = 1/(1 - E^2) \).

c) Find the quadratic approximation to \( 1/(1 - E^2) \) near \( E = 0 \). That is, find the constants \( A_0, A_1 \) and \( A_2 \) such that \( 1/(1 - E^2) = A_0 + A_1 E + A_2 E^2 + O(E^3) \) near \( E = 0 \).

d) Imagine that you have made a cubic approximation to \( \tanh^{-1}(E) \) near \( E = 0 \). That is, assume that you have approximated \( \tanh^{-1}(E) = B_0 + B_1 E + B_2 E^2 + B_3 E^3 + O(E^4) \) for constants \( B_0, B_1, B_2, B_3 \). Assume that \( B_0 + B_1 E + B_2 E^2 \) is the usual quadratic approximation to \( \tanh^{-1}(E) \) near \( E = 0 \). Assume in addition that the derivative of your cubic approximation to \( \tanh^{-1}(E) \) is precisely the quadratic approximation to \( 1/(1 - E^2) = (d/dE) \tanh^{-1}(E) \) that you found in part b). Under these assumptions, find the constants \( B_0, B_1, B_2, B_3 \).
e) Use part d) to find the *quadratic* approximation to $f(E)$ near $E = 0$.

f) Use part e) to find the quadratic approximation to $S$ near $E = 0$.

*Remark:* The quadratic approximation procedure outlined in this problem can be rigorously justified. It turns out that the quadratic approximations you are deriving are in some sense the “best possible ones.” These ideas fall under the umbrella of Taylor series, which we will discuss at the end of the course.

g) When $E$ is positive but near 0, does your quadratic approximation predict that the surface area of the ellipsoid will be greater than or less than the surface area of the perfect sphere of radius 1 (the perfect sphere has $E = 0$)?

3. (2 + 2 + 2 + 2 + 2 = 10 points) Section 4.1: 18abc, 20ab

4. (3 + 3 + 3 = 9 points) Section 4.2: 12, 16, 18

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