Indeterminate Forms: L'Hôpital's Rule

- Sometimes when we are computing limits, we encounter indeterminate expressions such as \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \).

\[
\text{Ex: } \lim_{x \to 1} \frac{x^3-1}{x^2-1} = \lim_{x \to 1} \frac{0}{0} = ??
\]

\[
\text{Ex: One way to deal with these difficulties is by using algebra to simplify the expressions.}
\]

\[
\lim_{x \to 1} \frac{x^3-1}{x^2-1} = \lim_{x \to 1} \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} = \lim_{x \to 1} \frac{x^2+x+1}{x+1} = \frac{3}{2}
\]

- Alternate approach:

- L'Hôpital's rule, easy version: If \( f(a) = g(a) = 0 \), then

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} \text{ as long as } g'(a) \neq 0.
\]

**Proof:**

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x)-f(a)}{g(x)-g(a)} \cdot \lim_{x \to a} \frac{x-a}{g(x)-g(a)} = \frac{f'(a)}{g'(a)}.
\]
In the previous example,

\[ f(x) = x^3 - 1, \quad g(x) = x^2 - 1, \quad f'(a) = g'(a) = 0 \]

\[ f'(x) = 3x^2, \quad g'(x) = 2x \]

\[ f'(a) = 3, \quad g'(a) = 2, \quad \frac{f'(a)}{g'(a)} = \frac{3}{2} \]

Ex: Let's apply L'Hôpital's rule to

\[ \lim_{x \to 1} \frac{x^{15} - 1}{x^3 - 1} \]

\[ \lim_{x \to 1} \frac{x^{15} - 1}{x^3 - 1} = \lim_{x \to 1} \frac{15x^{14}}{3x^2} = \frac{15}{3} = 5. \]

Alternate approach - linear approximation

\[ f(x) = x^{15} - 1, \quad a = 1, \quad f'(a) = 0, \quad m = f'(1) = 15 \]

\[ f(x) \approx m(x-a) + f(a) = 15(x-1) \]

Similarly, \[ g(x) = x^3 - 1 \approx 3(x-1) \]

Hence, \[ \frac{x^{15} - 1}{x^3 - 1} \approx \frac{15(x-1)}{3(x-1)} = 5 \]
Ex: Let's apply L'Hopital's rule to compute \( \lim_{x \to 0} \frac{\sin(3x)}{x} = \lim_{x \to 0} \frac{3\cos(3x)}{1} = 3 \).

This is the same as

\[
\frac{d}{dx} \sin(3x) \bigg|_{x=0} = 3\cos(3x) \bigg|_{x=0} = 3
\]

Ex: \( \lim_{x \to \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}} = \lim_{x \to \frac{\pi}{4}} \frac{\cos x + \sin x}{1} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} \)

Remark: Derivatives \( \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \) are always a \( \frac{0}{0} \) type limit.

Ex: \( \lim_{x \to 0} \frac{\cos x - 1}{x} = \lim_{x \to 0} \frac{-\sin x}{1} = 0 \)
\[ \lim_{x \to 0} \frac{\cos x - 1}{x^2} = \lim_{x \to 0} \frac{-\sin x}{2x} = \lim_{x \to 0} \frac{-\cos x}{2} = -\frac{1}{2} \]

- Alternate approach: quadratic approximation:

\[ \cos x \approx 1 - \frac{1}{2} x^2 \quad \text{when} \quad x \approx 0 \]

\[ \Rightarrow \quad \frac{\cos x - 1}{x^2} \approx \frac{(1 - \frac{1}{2} x^2) - 1}{x^2} = -\frac{1}{2} \]

\[ \lim_{x \to 0} \frac{\sin x}{x^2} = \lim_{x \to 0} \frac{\cos x}{2x} = \text{this limit does not exist!} \]

- Since \( \lim_{x \to 0} \frac{\cos x}{2x} \) is not of the form \( \frac{0}{0} \)\)

You cannot and should not apply L’Hôpital’s rule!
L'Hopital's rule can also be used on limits of the form \( \frac{\infty}{\infty} \) or if \( x \to \pm \infty \)

Let's figure out which function goes to \( \infty \) faster as \( x \to \infty \): \( x^n, e^{ax}, \) or \( \ln x \).

**Ex:** For \( a > 0 \)
\[
\lim_{x \to \infty} \frac{e^{ax}}{x} = \lim_{x \to \infty} \frac{ae^{ax}}{1} = +\infty
\]

Hence, when \( a > 0 \), \( e^{ax} \) grows faster than \( x \).

**Ex**
\[
\lim_{x \to \infty} \frac{e^{ax}}{x^{10}} = \lim_{x \to \infty} \frac{ae^{ax}}{10x^9} = \lim_{x \to \infty} \frac{a^2e^{ax}}{10 \cdot 9 \cdot x^8} = \ldots = \lim_{x \to \infty} \frac{a^{10}e^{ax}}{10!} = +\infty
\]

Alternate solution:
\[
\frac{e^{ax}}{x^{10}} = \left( \frac{e^{\frac{a}{10}x}}{x} \right)^{10}
\]

We have already shown that \( \frac{e^{\frac{a}{10}x}}{x} \to \infty \) as \( x \to \infty \).

Thus
\[
\lim_{x \to \infty} \frac{e^{ax}}{x^{10}} = \lim_{x \to \infty} \left( \frac{e^{\frac{a}{10}x}}{x} \right)^{10} = \infty^{10} = \infty.
\]
Ex \( \lim_{x \to 0^+} \frac{\ln x}{x^{\frac{1}{3}}} = \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{3}x^{-\frac{2}{3}}} = \lim_{x \to \infty} 3x^{\frac{1}{3}} = 0 \)

1. Combining the previous examples, we see that when \( a > 0 \) and \( x \to \infty \),

\[ \ln x \ll x^{\frac{1}{3}} \ll x \ll x^{10} \ll e^{ax} \]

2. L'Hôpital's rule directly applies to \( \frac{0}{0} \) and \( \frac{\infty}{\infty} \).

3. However, we sometimes encounter other indeterminate limits such as \( 0 \cdot \infty \), \( 0/0 \), and \( \infty/\infty \).

4. Using algebra, exponentials, and logs, we can put these other indeterminate limits into standard L'Hôpital form.
Ex. Let's calculate \( \lim_{x \to 0} x^x \) (\( 0^0 \) form)

- First rule: \( x^x = e^{\ln x^x} = e^{x \ln x} \)

- Let's calculate \( \lim_{x \to 0} x \ln x \) (\( 0 \cdot (\infty) \) form)

- We try putting it into \( \frac{0}{0} \) form:
  \[
  \frac{x}{x \ln x}
  \]

  However, since we don't know how to find \( \lim_{x \to 0} \frac{1}{x \ln x} \), this approach is not helpful.

- So we instead try to put it into \( \frac{\infty}{\infty} \) form:
  \[
  \frac{\ln x}{x}
  \]

- By L'Hôpital, we find that
  \[
  \lim_{x \to 0} x \ln x = \lim_{x \to 0} \frac{\ln x}{x} = \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0} -x = 0
  \]

- Thus, \( \lim_{x \to 0} x^x = \lim_{x \to 0} e^{x \ln x} = e^{\lim_{x \to 0} x \ln x} = e^0 = 1 \)

  Since \( e^u \) is a continuous function of \( u \)