The Polynomial Method, Fall 2012, Project List

Very optional questions you could explore. Generally I don’t know the answers.

1. Distinct lines. Suppose that we have $A$ distinct lines in $\mathbb{F}_q^n$. Let $X$ be the union of the lines. How small can $X$ be? Using a bush-type argument, prove that $|X| \gtrsim A^{1/2}q$. In some cases this is sharp. For example, if $A = q^2$, we get a lower bound of $q^2$. A plane has $q^2$ points in it and contains $q^2$ distinct lines. But what about $A = q^3$? The bush lower bound is now $|q|^{5/2}$, but there is no such thing as a $(5/2)$-dimensional plane. We can find $q^3$ (or even $q^4$) lines in a 3-dimensional plane, giving examples where $|X| = q^3$. Can you find a better example or prove a better lower bound?

2. Furstenberg-type problem in finite fields. (This is similar to a question that Nate mentioned to me.) Suppose that $X \subset \mathbb{F}_q^n$, and that for every direction, there is a line $l$ in the given direction so that $|X \cap l| \geq A$. In terms of $n$ and $A$, how small can $|X|$ be? If we take $A = q$, then $X$ is a Kakeya set, but it would be interesting to understand what happens for lower values of $A$ like $q^{1/2}$ or even 10.

3. Can you formulate and investigate a version of the Elekes-Sharir conjectures for lines in $\mathbb{R}^4$?

4. Can you prove a general version of Bezout’s theorem with an argument along the lines of the one in lecture 13.

5. Degree reduction in 5 dimensions? In the problem set, you will explore degree reduction for sets in 4 dimensions. There are some new issues that occur in 5 dimensions... Could be good for someone with some background in commutative algebra.