**Theorem 0.1** (3D Szemerédi-Trotter). Given $S$ points and $L$ lines in $\mathbb{R}^3$ with at most $B$ lines in any plane, the number of incidences $I(S, L)$ is at most $S^{\frac{2}{3}}L^{\frac{2}{3}} + B^{\frac{1}{3}}L^{\frac{2}{3}}S^{\frac{2}{3}} + S + L$.

The four terms of that sum are tight for, respectively, a 3-D grid, $L/B$ planes with $B$ lines in each with the 2-D Szemerédi-Trotter arrangement, all points collinear, and all lines concurrent, respectively.

We already know that $I(S, L) \leq S^2 + L$ and $I(S, L) \leq L^2 + S$ by counting, and $I(S, L) \leq C[L^{\frac{2}{3}}S^{\frac{2}{3}} + L + S]$ by Szemerédi-Trotter. So we’re already done unless $S \leq L^2 \leq S^4$ (ignoring constants).

**Claim 1** (Cell Estimate). In a polynomial cell decomposition of degree $d$, $I(S, L) \leq C[d^{-\frac{1}{3}}S^{\frac{2}{3}}L^{\frac{2}{3}} + dL + S_{\text{cell}}] + I(S_{\text{alg}}, L_{\text{alg}})$.

**Proof.** Let the cells be $O_i$, and let $S_i$ and $L_i$ be the number of points and lines that intersect $O_i$. Then $\sum S_i = S_{\text{cell}} \leq S$, $\sum L_i \leq dL$, and $S_i \leq d^{-\frac{3}{2}}S$. (Here and henceforth, we drop constants.)

Then $I(S_{\text{cell}}, L) = \sum_i I(S_i, L_i) \leq \sum_i S_i^\frac{2}{3}L_i^{\frac{2}{3}} + L_i + S_i \leq (d^{-1}S^{\frac{1}{3}}\sum_i S_i^\frac{1}{3}L_i^{\frac{2}{3}}) + \sum_i L_i + S_i$. By Hölder’s inequality, that’s at most $(d^{-1}S^{\frac{1}{3}}(\sum_i S_i)^\frac{1}{3})(\sum L_i)^{\frac{2}{3}} + \sum_i L_i + S_i = d^{-\frac{1}{3}}S^{\frac{2}{3}}L^{\frac{2}{3}} + dL + S_{\text{cell}}$.

Finally, $I(S_{\text{alg}}, L_{\text{cell}}) \leq dL$ by degree bounding, so we’ve counted everything but $I(S_{\text{alg}}, L_{\text{alg}})$, as desired. \[\square\]

Let $L_p$, $L_m$ and $L_u$ (“planar,” “multiplanar,” and “uniplanar”) be the sets of lines in at least one, at least two, and exactly one plane of $Z(P)$, respectively, and let $S_p$, $S_m$, and $S_u$ be the same for points.

**Claim 2** (Planar Estimate). $I(S_{\text{alg}}, L_p) \leq C[B^{\frac{1}{3}}L^{\frac{1}{3}}S^{\frac{2}{3}} + dL + S_u] + I(S_m, L_m)$.

Also, $|L_m| \leq d^2$; we’ll choose $d$ small enough that the last term is handleable by induction.

**Proof.** $I(S_{\text{alg}}, L_p) \leq I(S_{\text{alg}}, L_u) + I(S_m, L_m)$, since a line in multiple planes only hits points in multiple planes. Let $P$ be the set of planes in $Z(P)$.

$I(S_{\text{alg}}, L_u) \leq \sum_{\pi \in P} I(S_{\pi}, L_{u,\pi}) \leq \sum_{\pi} dL_{u,\pi} + I(S_{u,\pi}, L_{u,\pi})$. By the same application of Hölder’s Inequality as before, that’s at most $dL + B^{\frac{1}{3}}L^{\frac{1}{3}}S^{\frac{2}{3}} + S_u$. \[\square\]

That leaves the nonplanar algebraic lines (and multiplanar lines) to bound. We’ll use special points, that is, flat or critical points, that is, points at which $SP$ (which has degree at most $3d$) is 0 and special lines, on which every point is special.

Let $S_s$ and $S_n$ be the sets of special and nonspecial points, respectively, in $S_{\text{alg}}$, and define $L_s$ and $L_n$ similarly.

**Claim 3** (Algebraic Estimate). $I(S_{\text{alg}}, L_{\text{alg}} \setminus L_p) \leq C[dL + S_n] + I(S_s, L_{s \setminus L_p})$, and $|L_{s \setminus L_p}| \leq 10d^2$.

**Proof.** Recall that

1. If $x$ is in three lines of $Z(P)$ then $x$ is special,
2. \( x \) is special iff \( SP(x) = 0 \), where \( \deg(SP) \leq 3d \), and

3. The number of lines that are special but not planar is at most \( 10d^2 \).

Now, \( I(S_{\text{alg}}, L_{\text{alg}} \setminus L_p) \leq I(S_n, L_{\text{alg}}) + I(S_s, L_{\text{ng}}) + I(S_s, L_s \setminus L_p) \). The first term is at most \( 2S_n \) by the first recalled property and the second term is at most \( 3dL \) by the second recalled property.

That leaves \( I(S_s, L_s \setminus L_p) \) and \( I(S_m, L_m) \) to bound; those contain at most \( 11d^2 \) lines. Let \( S' = S \setminus (S_s \cup S_m) \). We already have \( I(S, L) \leq d^{-\frac{1}{3}}L^{\frac{2}{3}}S^{\frac{2}{3}} + dL + B^{\frac{1}{3}}L^{\frac{2}{3}}S^{\frac{2}{3}} + S' + I(S_s, L_s \setminus L_p)I(S_m, L_m) \).

**Lemma 1.** The minimum value of \( d^{-\frac{1}{3}}L^{\frac{2}{3}}S^{\frac{2}{3}} + dL + B^{\frac{1}{3}}L^{\frac{2}{3}}S^{\frac{2}{3}} + S \) with \( d \in [1, \frac{1}{2}L^{\frac{2}{3}}] \) (and \( B \geq L^{\frac{1}{2}} \)) is about \( S^{\frac{1}{2}}L^{\frac{1}{3}} + B^{\frac{1}{3}}L^{\frac{2}{3}}S^{\frac{2}{3}} + S' \).

**Proof.** Just do it.

So \( I(S, L) \leq C[S^{\frac{1}{2}}L^{\frac{1}{3}} + B^{\frac{1}{3}}L^{\frac{2}{3}}S^{\frac{2}{3}} + S'] + C_0[S^{\frac{1}{2}}(\frac{L}{2})^{\frac{1}{3}} + B^{\frac{1}{3}}(\frac{L}{2})^{\frac{2}{3}}S^{\frac{2}{3}} + (S - S')] \), and we can choose \( C_0 \) arbitrarily and bigger than, say, \( 100C \), so that’s at most \( C_0[S^{\frac{1}{2}}L^{\frac{1}{3}} + B^{\frac{1}{3}}L^{\frac{2}{3}}S^{\frac{2}{3}} + S + L] \), as desired.

### 0.1 Efficiency of Polynomials

**Theorem 0.2** ("Efficiency of Polynomials"). If \( P : \mathbb{C} \to \mathbb{C} \) is a polynomial and \( F : \mathbb{C} \to \mathbb{C} \) is smooth (not necessarily holomorphic), and \( F = P \) outside some bounded domain \( \Omega \), and \( 0 \) is a regular value of \( P \) and \( F \), then \( P \) has at most as many zeros in \( \Omega \) as \( F \) does.

(If \( F : M^m \to N^n \) is a function, then \( x \in M \) is a critical point iff \( dF_x \) isn’t surjective, and a regular point otherwise. \( y \in N \) is regular iff all its preimages are regular. In our case, if \( x \in Z(F) \), that \( 0 \) is a regular value implies that \( dF_x : \mathbb{R}^2 \to \mathbb{R}^2 \) is an isomorphism. Call \( \sigma(x) = 1 \) if \( dF_x \) preserves orientation and -1 otherwise.)

If \( P \) is a complex polynomial, then \( \sigma_P(x) = +1 \) for all \( x \in Z(P) \).

**Theorem 0.3.** The winding number of \( F : \partial \Omega \to \mathbb{C} \setminus \{0\} \) is \( \sum_{x \in Z(F) \cap \partial \Omega} \sigma_F(x) = \sum_{x \in Z(P) \cap \partial \Omega} \sigma_P(x) \).