Real analysis, Optional problems and projects

If you’re interested in further exploring some of the topics in the course, here are some problems you could think about. They are a little meatier than the problems on the recent problem sets, but they are also approachable. I would be happy to talk about them, and if you write up a solution or discussion, I would be happy to read it.

1. (Exploring linear operators) Let $T$ be a linear operator. Define $E(T)$ to be the set of pairs $(p,q) \in [1,\infty]^2$ so that $\|Tf\|_q \lesssim \|f\|_p$. Define $IE(T)$ to be the inverses, the set of pairs $(1/p,1/q)$ so that $\|Tf\|_q \lesssim \|f\|_p$. The interpolation theorem implies that $IE(T)$ is a convex subset of $[0,1]^2$. Which convex subsets could it be?

   a.) Given a pair $(p,q)$, can you find a linear operator $T$ so that $E(T)$ is exactly the point $(p,q)$?

   b.) Given $(p_0,q_0)$ and $(p_1,q_1)$, can you find a linear operator $T$ so that $IE(T)$ is exactly the
   closed line segment from $(1/p_0,1/q_0)$ to $(1/p_1,1/q_1)$?

   c.) Given a convex set $K \subset [0,1]^2$, can you find a linear operator $T$ so that $IE(T) = K$?

2. (Using the Calderon-Zygmund theorem) The Calderon-Zygmund inequality has important applications in elliptic PDE. Here is one example. The proof involves CZ and some ideas from the elliptic section of our course.

   **Theorem 1.** For any dimension $n$, there is a constant $\epsilon(n) > 0$ so that the following holds. Suppose that $|a_{ij} - \delta_{ij}| < \epsilon(n)$, and let $Lu = \sum_{ij} a_{ij} \partial_i \partial_j u$. Suppose that $Lu = 0$ on $B_1$. Then for any $\alpha < 1$, the following estimate holds:

   $$\|u\|_{C^{1,\alpha}(B_1/2)} \leq C(\alpha,n)\|u\|_{C^{0}(B_1)}.$$  

3. (Decay of solutions to the wave equation) Suppose that $u$ is a solution of the wave equation on $\mathbb{R}^n \times \mathbb{R}$ with initial data $u[0] = (f,g)$. Suppose that $f,g$ are supported in the unit ball with $\|f,g\|_{C^{n+1}} \leq 1$. Prove the following decay estimate for $u$:

   $$|u(x,t)| \leq C_n(1 + |t|)^{-\frac{n+1}{2}}.$$  

   We discussed how to approach this problem in class, using the Fourier transform and some difficult integration by parts. (You might also want to look up ‘stationary phase’ for a description of how this works.) To keep the computations simpler, you may want to assume $g = 0.$