1 Classical Approach

Our goal in these notes will be to prove the following theorem:

**Theorem 1.1** (DiGeorgi-Nash-Moser). Let

\[ Lu := \sum \partial_i (a_{ij} \partial_j u) \text{ and } 0 < \lambda \leq a_{ij} \leq \Lambda \]

Then there exists \( \alpha(n, \lambda, \Lambda) > 0 \) and \( C(n, \lambda, \Lambda) \) such that if \( Lu = 0 \), then

\[ \| u \|_{C^\alpha(B_{1/2})} \leq C(\lambda, \Lambda, n) \| u \|_{C^0(B_1)} \]

Note that this estimate does not in any way involve derivatives of the \( a_{ij} \).

We start by reminding of the Dirichlet energy of a function:

**Definition 1.1** (Dirichlet Energy). If \( u : \Omega \rightarrow \mathbb{R} \), then \( E(u) = \int_\Omega |\nabla u|^2 \).

With this, we have the following easy proposition.

**Proposition 1.1.** If \( u, w \in C^2(\bar{\Omega}) \), \( u = w \) on \( \partial \Omega \), and \( \Delta u = 0 \), then \( E(u) \leq E(w) \).
Proof. : Let $w = u + v$, so $v|_{\partial \Omega} = 0$. Then

$$E(w) = \int_\Omega \langle \nabla w, \nabla w \rangle = \int_\Omega |\nabla u|^2 + |\nabla v|^2 + 2 \int_\Omega \nabla u \cdot \nabla v$$

$$\leq \int_\Omega |\nabla u|^2 = E(u)$$

where we got from the first line to the second by integration by parts. \qed

In a similar way, we can define

**Definition 1.2** (Gen. Dirichlet Energy). If $L, a$ satisfies (DGH), then

$$E_a(u) = \int_\Omega \sum a_{ij} (\partial_i u)(\partial_j u)$$

and get a similar proposition with identical proof:

**Proposition 1.2.** If $u, w \in C^2(\bar{\Omega})$, and $u = w$ on $\partial \Omega$, and $Lu = 0$, then $E_a(w) \geq E_a(u)$.

We now prove an $L^2$ estimate relating $\nabla u$ to $u$.

**Proposition 1.3.** If $L$ follows (DGH) and $Lu = 0$ on $B_1$ then

$$\int_{B_{1/2}} |\nabla u|^2 \lesssim \int_{B_1} |u|^2$$

**Proof.** We will use integration by parts and localization. Let $\eta = 1$ on $B_{1/2}$ and be 0 outside of $B_1$.

$$\int_{B_{1/2}} |\nabla u|^2 \leq \int \eta^2 |\nabla u|^2 \approx \int \eta^2 \sum a_{ij} \partial_i u \partial_j u$$

$$\leq \int \eta^2 (Lu) u + \int |\nabla \eta| |\nabla u||u|$$

$$\leq \left( \int \eta^2 |\nabla u|^2 \right)^{1/2} \left( \int |\nabla \eta|^2 u^2 \right)^{1/2}$$

\qed
A classical approach would be to then prove the following:

**Proposition 1.4.** If \((DGH), Lu = 0\) and \(\|a_{ij}\|_{C^1} \leq B\) then

\[
\int_{B_{1/2}} |D^2 u|^2 \leq C(B, n, \lambda, \Lambda) \int_{B_{3/4}} |\nabla u|^2
\]

**Proof.** We have that \(0 = \partial_k Lu = L(\partial_k u) + (\partial_k a_{ij})\partial_i \partial_j u\). Then,

\[
\int_{B_{1/2}} |D^2 u|^2 \leq \int \eta^2 \sum a_{ij} \partial_i \partial_k u \partial_j \partial_k u
\]

\[
\leq \int |\nabla \eta| |\eta| D^2 u |\nabla u| + \int \eta^2 L(\partial_k u) \partial_k u
\]

\[
\leq \int |\nabla \eta| |\eta| D^2 u |\nabla u| + \int \eta^2 B |D^2 u| |\nabla u|
\]

The result comes from applying Cauchy-Schwartz to this last pair of terms. \(\square\)

However, this won’t get us closer to proving DiGeorgi-Nash-Moser because we’re using an estimate on the derivatives of \(a\) in our inequality. Looks like we’ll have to be clever!

## 2 \(L^\infty\) Bound

**Theorem 2.1** (DGNM \(L^\infty\) bound). Let \(L\) satisfy \((DGH), Lu \geq 0, u > 0\). Then

\[\|u\|_{L^\infty(B_{1/2})} \leq \|u\|_{L^2(B_1)}\]

**Proof.** We start with a lemma:

**Lemma 2.1.** Under the hypotheses, and if \(1/2 \leq r < r + w \leq 1\) then

\[\|\nabla u\|_{L^2(B_r)} \lesssim \|u\|_{L^2(B_{r+w})} w^{-1}\]

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Proof. Let $\eta = 1$ on $B_r$ and 0 on $B_{r+w}^c$. Note that $\eta$ can be constructed so that $|\nabla \eta| < 2w^{-1}$. Then the proof proceeds in exactly the same fashion as Proposition 1.3.

**Lemma 2.2.** Under hypotheses, and $1/2 \leq r < r + 2 \leq 1$, we have

$$\|u\|_{L^{2n/(n-2)}(B_r)} \lesssim w^{-1}\|u\|_{L^2(B_{r+w})}$$

*Proof.* Consider $\eta u$ with $\eta = 1$ on $B_r$, and 0 outside of $B_{r+w}/2$. Then by the Sobolev inequality, we have

$$\|\eta u\|_{L^{2n/(n-2)}} \lesssim \|\nabla (\eta u)\|_{L^2} \leq \|\nabla \eta\|_{L^2} + \|\eta(u)\|_{L^2}$$

Also, we have that

$$\|\nabla \eta\|_{L^2} \leq \|\nabla \eta\|_{L^\infty} \|u\|_{L^2(B_{r+w})} \lesssim w^{-1}\|u\|_{L^2(B_{r+w})}$$

$$\|\eta(u)\|_{L^2} \leq \|\nabla u\|_{L^2(B_{r+w})} \lesssim w^{-1}\|u\|_{L^2(B_{r+w})}$$

□

**Lemma 2.3.** If $\beta > 1$, $Lu \geq 0$ and $u > 0$, then $Lu^\beta \geq 0$.

*Proof.* Compute:

$$Lu^\beta = \sum \partial_i (a_{ij} \partial_j (u^\beta)) = \sum \partial_i (a_{ij} \beta u^{\beta-1} \partial_j u)$$

$$= (Lu)(\beta u^{\beta-1}) + \sum a_{ij} \partial_i u \partial_j u \beta (\beta - 1) u^{\beta-2} \geq 0$$

where the last inequality comes from ellipticity of $a_{ij}$. □

Now, apply Lemma 2.2 to $u^\beta$ to get

$$\|u^\beta\|_{L^{2n/(n-2)}(B_r)} \lesssim w^{-1}\|u^\beta\|_{L^2(B_{r+w})}$$

Rewriting this with $s = \frac{n}{n-2}$ we get
Lemma 2.4. If $1/2 \leq r < r + w \leq 1$ and $p \geq 2$, then
\[
\|u\|_{L^p(B_r)} \leq (Cw^{-1})^{2/p}\|u\|_{L^p(B_{r+w})}
\]

For the next step, we iterate this lemma. If we have $1 = r_0 > r_1 > \cdots > r_k > 1/2$, then we get the sequence of inequalities
\[
\|u\|_{L^2(B_{r_j})} \geq A_0\|u\|_{L^{2s}(B_{r_j})} \geq \cdots \geq A_0 \cdots A_{k-1}\|u\|_{L^{2s}(B_{r_k})}
\]
where the $A_j$ are given by Lemma 2.4. Let’s pick $r_j = \frac{1}{2} + \frac{1}{j+2}$, so that $r_j - r_{j+1} \approx j^{-2}$. Thus, $A_j = (C(r_j - r_{j-1})^{-1})^{s^{-j}}$. Therefore,
\[
\log(\prod A_j) \leq \sum \log(A_j) \leq \sum_{j=0}^\infty s^{-j}(C + C\log(r_j - r_{j+1}))
\]
\[
\leq \sum_{j=0}^\infty s^{-j}(C + C\log j) < \infty
\]

3 Finishing the Proof

Recall the Harnack inequality:

**Theorem 3.1** (Harnack). If $\Delta u = 0$ on $B_1$ and $u > 0$ then $\min_{B_{1/2}} u \geq \gamma(n) \max_{B_1} u$.

We will show a Harnack inequality for our $L$ which satisfies (DGH).

**Theorem 3.2** (DGNM Harnack). If $L$ satisfies (DGH), $Lu = 0$, $1 > u > 0$ on $B_1$, and
\[
|\{x \in B_{1/2} | u(x) > 1/10\}| \geq \frac{1}{10}|B_{1/2}|
\]
then $\min_{B_{1/2}} u \geq \gamma(n)$.

For now, let’s assume this theorem, and see how it implies the DiGiorgi-Nash-Moser estimate.
Definition 3.1. $\text{osc}_\Omega u := \sup_\Omega u - \inf_\Omega u$.

Corollary 3.1. If $Lu = 0$ on $\Omega$, $B_r(x) \subset \Omega$, then
\[ \text{osc}_{B_r(x)} u \leq (1 - \gamma)\text{osc}_{B_r(x)} u \quad (O) \]

Proof. We start with some simple reductions via scaling. Without loss of generality, we can take:
\[ \inf_{B_r(x)} u = 0, \quad \sup_{B_r(x)} u = 1, \quad r = 1 \]
\[ |\{ x \in B_{1/2} | u(x) \geq 1/2 \}| \geq B_{1/2}/2 \]

Thus by DGNM Harnack, $\min_{B_{1/2}} u \geq \gamma$, and thus $\text{osc}_{B_{1/2}} u \leq 1 - \gamma = (1 - \gamma)\text{osc}_{B_1} u$.

Now we can complete the proof with the following:

Proposition 3.1. Let $u : B_1 \to \mathbb{R}$ satisfy (O). Then $\| u \|_{C^0(B_{1/2})} \lesssim \| u \|_{C^0(B_1)}$ for some $\alpha = \alpha(\gamma) > 0$.

Proof. Let $x, y \in B^{1/2}$, $|x - y| = d$ and $a = (x + y)/2$. Then
\[ |u(x) - u(y)| \leq (\text{osc}_{B_d(a)} u)(1 - \gamma) \leq \cdots \leq (1 - \gamma)^k \text{osc}_{B_{d^k(a)}} u \]

Choose $k$ such that $1/4 < 2^kd \leq 1/2$. Then $k = \log_2(1/d) + O(1)$, and so
\[ |u(x) - u(y)| \leq (1 - \gamma)^k \text{osc}_{B_1} u \leq 2(1 - \gamma)^k \| u \|_{C^0(B_1)}. \]

Also,
\[ (1 - \gamma)^k \leq 4(1 - \gamma)^{\log_2(1/d)} = 4d^{-\log_2(1-\gamma)}. \]

Therefore, setting $\alpha(\gamma) = -\log_2(1 - \gamma) \approx \gamma + O(\gamma^2)$, we get our proposition.

Now let’s prove the Harnack inequality. Before we do the DGNM Har-
nack, we’ll remember how the normal $\Delta$ Harnack inequality works:

Lemma 3.1. If $\Delta u = 0$ and $u > 0$ then $\| \nabla \log u \|_{L^\infty(B_{1/2})} \lesssim 1$. 

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Note that the lemma implies the Harnack inequality by integrating.

**Proof.** We have \( \nabla \log u = \frac{\nabla u}{u} \). Also, by elliptic regularity, we have that

\[
|\nabla u|(x) \lesssim \|u\|_{L^1(B_{1/2}(x))} = \int_{B_{1/2}(x)} u = |B_{1/2}(x)|u(x)
\]

so that \( |\nabla u|/u \lesssim 1 \). \( \square \)

With this method in mind, let’s prove the DGNM Harnack.

**DGNM Harnack.**

**Lemma 3.2.** If \( L \) satisfies (DGH), \( Lu = 0, u > 0 \) on \( B_1 \) then \( \|\nabla \log u\|_{L^2(B_{1/2})} \lesssim 1 \).

**Proof.** Pick a nice cutoff function \( \eta \) as usual.

\[
\int_{B_{1/2}} |\nabla \log u|^2 = \int \eta^2 |\nabla \log u|^2 \lesssim \int \eta^2 \sum a_{ij} \partial_i \log u \partial_j \log u
\]

\[
= \int \eta^2 \sum a_{ij} \frac{\partial_i u \partial_j u}{u} = -\int \eta^2 \sum a_{ij} \partial_i u \partial_j u^{-1}
\]

\[
\lesssim \int \eta |\nabla \eta||\nabla u||u^{-1} = \int \eta |\nabla \eta||\nabla \log u|
\]

\[
\leq \left( \int \eta^2 |\nabla \log u|^2 \right)^{1/2} \left( \int |\nabla \eta|^2 \right)^{1/2}
\]

\( \square \)

Letting \( w = -\log u \), we have that \( \|\nabla w\|_{L^2(B_{9/10})} \lesssim 1 \). We want an \( L^\infty \) bound on \( w \). By (P), we have that

\[
|\{ x \in B_{1/2}| w \leq \log 10 \}| \geq \frac{1}{10} |B_{1/2}|
\]

Now we use the Poincare Inequality:
**Theorem 3.3** (Poincare). If (P) then \( \int_{B_{8/10}} |w|^2 \lesssim \int_{B_{9/10}} |
abla w|^2 + 1 \)

Therefore, we have an \( L^2 \) bound on \( w \) instead of \( \nabla w \). Now we have

**Lemma 3.3.** \( \mathcal{L}w \geq 0 \)

**Proof.** Compute:

\[
- \sum \partial_i (a_{ij} \partial_j \log u) = - \sum \partial_i (a_{ij} (\partial_j u) u^{-1}) \\
= \mathcal{L}u \cdot u^{-1} + \sum a_{ij} (\partial_i u)(\partial_j u) u^{-2} \geq 0
\]

\( \square \)

Finally, \( w = - \log u > 0 \) because \( u < 1 \), and so we can apply Theorem 2.1 and get

\[
\|w\|_{L^\infty(B_{1/2})} \lesssim \|w\|_{L^2(B_{8/10})} \lesssim 1
\]

thus completing the proof of the Harnack inequality.

\( \square \)