QUASI-ISOGENIES OF p-DIVISIBLE GROUPS

VIJAY SRINIVASAN

Mostly I will follow the section from [RZ] §2.1-§2.12 but there are several added details. The talk will have three technical goals:

- develop background on formal schemes,
- collect properties of quasi-isogenies of p-divisible groups, and
- define the moduli functor of quasi-isogenies.

Next week's talk will cover the *representability* of the moduli functor.

1. Review of formal schemes

Definition 1.1. Let $(A, \{I_\alpha\})$ be a topological ring together with a set of ideals that form a fundamental system of neighborhoods of 0. We say that A is:

- *pre-admissible* if there is an open ideal I of A (called an *ideal of definition*) such that each I_α contains some power of I,
- *pre-adic* if it is pre-admissible and the ideal I can be chosen so that $\{I^n\}$ forms a fundamental system of neighborhoods of 0,
- *admissible* if it is pre-admissible and complete,
- *adic* if it is pre-adic and complete.

Example 1.2. If k is a perfect field, then $W(k)$ is adic with ideal of definition (p). If k is not perfect, then $W(k)$ is admissible but not adic (it is no longer true that $W_n(k) \cong W(k)/(p^n)$).

Proposition 1.3. *Let A be a preadmissible ring with fundamental system of ideal neighborhoods $\mathcal{I}_1 \supseteq \mathcal{I}_2 \supseteq \cdots$ and ideal of definition $\mathbf{I}$. Suppose the following hold:*

- *For every r, $\mathbf{I}/(\mathbf{I}^2 + \mathcal{I}_r)$ is of finite type, and*
- *For every m, the descending chain*
$$\mathcal{I}_1 + \mathbf{I}^m \supseteq \mathcal{I}_2 + \mathbf{I}^m \supseteq \cdots$$
stabilizes.

Then the completion of A is adic.

The above proposition may seem random, but it will be important in describing the formal scheme representing the moduli functor.

Definition 1.4. Let $(A, \{I_\alpha\})$ be a preadmissible ring. Define the functor $\operatorname{Spf} A : \mathrm{Sch}^{\mathrm{opp}} \to \mathrm{Set}$ via
$$\operatorname{Spf} A(Z) := \varinjlim_\alpha \operatorname{Hom}(Z, \operatorname{Spec} A/I_\alpha)$$
This is a Zariski sheaf when restricted to the qcqs schemes (but not for all schemes since colimit doesn't commute with arbitrary products). If A is adic, we say $\operatorname{Spf} A$ is an *affine formal scheme*. If $\mathscr{F}$ is a Zariski sheaf on the qcqs schemes, we call $\mathscr{F}$ a *formal scheme* if it has an open covering by affine formal schemes.

Lemma 1.5. *Suppose A is pre-adic. Then*

- *it is equivalent to write*
$$\operatorname{Spf} A(Z) = \varinjlim_n \operatorname{Hom}(Z, \operatorname{Spec} A/I^n)$$
- $\operatorname{Spf} A$ *defines the same functor on* $\mathrm{Sch}^{\mathrm{opp}}$ *as the locally ringed space*
$$(|\operatorname{Spec} A/I|, \varprojlim_n \mathscr{O}_{\operatorname{Spec} A}/\mathscr{I}^n).$$

Definition 1.6. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of formal schemes. We say that f is *of finite type* (resp. *étale*, resp. *smooth*) if for any scheme Z and morphism $Z \to \mathcal{Y}$, we have $\mathcal{X} \times_{\mathcal{Y}} Z$ is representable by a scheme and $\mathcal{X} \times_{\mathcal{Y}} Z \to Z$ is *of finite type* (resp. *étale*, resp. *smooth*).

Lemma 1.7. *For any formal scheme $\mathcal{X}$, there is a reduced scheme $\mathcal{X}_{\mathrm{red}}$ with a morphism $\mathcal{X}_{\mathrm{red}} \to \mathcal{X}$ such that the natural map*
$$\operatorname{Hom}(Z, \mathcal{X}_{\mathrm{red}}) \to \operatorname{Hom}(Z, \mathcal{X})$$
is bijective for any reduced scheme Z.

Proof. If $\mathcal{X} = \operatorname{Spf} A$ where I is a radical ideal of definition, set $\mathcal{X}_{\mathrm{red}} := \operatorname{Spec} A/I$. Globally, patch these together. □

Definition 1.8. A formal scheme $\mathcal{X}$ is called *locally Noetherian* if it is locally isomorphic to $\operatorname{Spf} A$ where A is an adic noetherian ring. A morphism $\mathcal{X} \to \mathcal{Y}$ of locally noetherian schemes is called *formally (locally) of finite type* if $\mathcal{X}_{\mathrm{red}} \to \mathcal{Y}_{\mathrm{red}}$ is (locally) of finite type.

Remark. The condition "of finite type" is very strict, because it requires all pullbacks by schemes to be representable. The condition "formally of finite type" is more relaxed; for example $\operatorname{Spf} k[[x]] \to \operatorname{Spec} k$ is formally of finite type but not of finite type.

2. Isogenies and quasi-isogenies

Definition 2.1. An *isogeny* $f : X \to Y$ of p-divisible groups over a scheme S is an epimorphism in the category of S-groups such that $\ker f$ is a finite locally free S-group scheme.

Proposition 2.2. *Let X be a p-divisible group over a connected scheme S. Then every finite locally free S-subgroup scheme of X is the kernel of an isogeny out of X.*

Proof. Let H be a finite locally free S-group representable by a scheme and $H \hookrightarrow X$ a monomorphism in the category of S-groups. We want to show that X/H is a p-divisible group. It is automatic that p is an epimorphism, so it is enough to check that $(X/H)[k]$ is finite locally free for every k.

Let H have order p^n. Then an argument of Deligne shows that H is killed by p^n, so $H \hookrightarrow X[n]$. For any $m \geq n$ we have an exact sequence
$$0 \longrightarrow X[m]/H \longrightarrow (X/H)[m] \longrightarrow H \longrightarrow 0$$
where the last map is multiplication by p^m. Observe that $X[m]/H$ is finite locally free since it is a quotient of finite locally free groups. Then $(X/H)[m]$ is finite locally free, being an extension of finite locally free groups. Finally for any k, we write
$$0 \longrightarrow (X/H)[n] \longrightarrow (X/H)[n+k] \longrightarrow (X/H)[k] \longrightarrow 0$$
which implies that $(X/H)[k]$ is finite locally free. □

Let X and Y be p-divisible groups over a base scheme S. Consider the Zariski sheaf $\underline{\mathrm{Hom}}_{S\text{-grp}}(X,Y)$. Observe that this is a $\mathbb{Z}_p$-module (either by precomposition or by post-composition; these are the same since we are considering homomorphisms of groups). It is torsion-free as a $\mathbb{Z}_p$-module because $[p]$ is an epimorphism on either X or Y.

Definition 2.3. A *quasi-isogeny* between X and Y is an element

$$\alpha \in \Gamma(S, \underline{\mathrm{Hom}}_{S\text{-grp}}(X,Y) \otimes \mathbb{Q})$$

such that for every point $s \in S$, there is a Zariski neighborhood $U \ni s$ and an integer n for which $(p^n\alpha)|_U$ is an isogeny. We write $\mathrm{Qisg}_S(X,Y)$ for the set of quasi-isogenies.

Lemma 2.4. *Quasi-isogenies admit quasi-inverses, that is, for any $\alpha \in \mathrm{Qisg}_S(X,Y)$, there is $\beta \in \mathrm{Qisg}(Y,X)$ such that $\beta \circ \alpha = \mathrm{id}_X$.*

Proof sketch. We give an informal argument on the level of points. It's enough to show that if $f : X \to Y$ is an isogeny, there is an isogeny $g : Y \to X$ such that $f \circ g = [p^n]$ for some n. We can choose n such that $\ker f \subseteq X[n]$. Then define $g : Y \to X$ via $g(y) = [p^n]f^{-1}(y)$, which is well-defined since elements of $f^{-1}(y)$ differ by elements of $X[n]$. Finally, $\ker g$ fits into the exact sequence

$$0 \longrightarrow \ker f \longrightarrow X[n] \longrightarrow \ker g \longrightarrow 0$$

and so $\ker g$ is finite locally free. □

Corollary 2.5. *Suppose $\alpha \in \mathrm{Qisg}_S(X,Y)$ and $p^n\alpha$ is an isogeny. Then α itself is an isogeny iff $(p^n\alpha)|_{X[n]} = 0$.*

Proposition 2.6 (Drinfeld rigidity property)**.** *Assume p is locally nilpotent on S. Let $\overline{S}$ be a closed subscheme of S cut out by locally nilpotent sheaf of ideals $\mathcal{I}$. Then the natural map $\mathrm{Qisg}_S(X,Y) \to \mathrm{Qisg}_{\overline{S}}(X_{\overline{S}}, Y_{\overline{S}})$ is a bijection.*

I won't prove this here; for a full proof see Andrè's book "Period mappings and differential equations" Theorem 2.2.3 and for the needed background on formal Lie groups see Katz's article "Serre-Tate local moduli." To show this map is a bijection, one must make use of the fact that p-divisible groups are automatically formally smooth when p is locally nilpotent on the base scheme S.

Lemma 2.7. *Let $\alpha : X \to Y$ be a quasi-isogeny. Then the functor $F : \mathrm{Sch}^{\mathrm{opp}} \to \mathrm{Set}$ given by*

$$F(T) = \{\phi \in \mathrm{Hom}(T,S) \mid \phi^*\alpha \textit{ is an isogeny}\}$$

is representable by a closed subscheme of S.

Proof. The condition that a homomorphism be an isogeny can be checked Zariski locally. So it suffices to consider the case where $p^n\alpha$ is an isogeny for a fixed n. Now

$$\phi^*\alpha \text{ is an isogeny} \iff \phi^*(p^n\alpha) \text{ kills } \phi^*X(n)$$

as a consequence of Corollary 2.5. Now view $p^n\alpha$ as a global section of $\underline{\mathrm{Hom}}_{\mathcal{O}_S}(X(n), Y(n))$, with zero locus Z. Then $\phi^*(p^n\alpha|_{X(n)}) = 0$ iff ϕ factors through Z. □

3. Defining the moduli functor

We will digress a bit and discuss isocrystals. Throughout, let L be a perfect field, $W := W(L)$ its Witt vectors, and $K_0 := W[\frac{1}{p}]$ the fraction field.

Definition 3.1. An isocrystal $(N, \mathbf{F})$ is called *decent* if it is spanned as a K_0-vector space by elements n satisfying $\mathbf{F}^s n = p^r n$ for some $r, s > 0$ (allowed to vary over different n).

We say a p-divisible group $\mathbb{X}$ over L is *decent* if its associated isocrystal is decent.

Observe that as a consequence of the slope decomposition, every isocrystal over an algebraically closed field is decent. Apparently, we will care most about the case $L = \overline{\mathbb{F}}_p$; when we specialize to this case we will not have to worry about decency.

Recall that an object $S \in \mathrm{Nilp}_W$ is a formal scheme over $\operatorname{Spf} W$. We write $\overline{S}$ for the subscheme cut out by the ideal sheaf $p\mathcal{O}_S$.

Definition 3.2. Fix a decent p-divisible $\mathbb{X}$ over L. We define the functor $\mathcal{M} : \mathrm{Nilp}_W \to \mathrm{Set}$ via so that $\mathcal{M}(S)$ consists of the set of pairs $\{(X, \rho)\}/\sim$, where X is a p-divisible group over S and $\rho \in \mathrm{Qisg}_{\overline{S}}(\mathbb{X}_{\overline{S}}, X_{\overline{S}})$, where $(X, \rho) \sim (X', \rho')$ iff there is an isomorphism $f : X \to X'$ making the following diagram commute

$$\begin{array}{ccc} \mathbb{X}_{\overline{S}} & = & \mathbb{X}_{\overline{S}} \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho'} \\ X_{\overline{S}} & \xrightarrow[f_{\overline{S}}]{} & X'_{\overline{S}} \end{array}$$

Remark. Drinfeld rigidity implies that a pair (X, ρ) does not have any automorphisms; indeed if $X = X'$ and $\rho = \rho'$ in the diagram above, then $f_{\overline{S}}$ is the identity map and so must lift to the identity map on X. A p-divisible group X without the extra data of ρ, however, possesses many nontrivial automorphisms (e.g. any element of $\mathbb{Z}_p^\times$).

We can give an alternative description of this moduli functor. By G–M deformation theory, we can choose a lift $\widetilde{\mathbb{X}}$ over $\operatorname{Spf} W$ such that the special fiber of $\widetilde{\mathbb{X}}$ is $\mathbb{X}$. Then by Drinfeld rigidity, $\mathcal{M}(S)$ is given by the set of pairs $\{(X, \tilde{\rho})\}/\sim$ where X is a p-divisible group over S and $\tilde{\rho} \in \mathrm{Qisg}_S(\widetilde{\mathbb{X}}_S, X)$.

Theorem 3.3. *The functor $\mathcal{M}$ is representable by a formal scheme, locally formally of finite type over* $\operatorname{Spf} W$.

This is the result that will be the focus of the next talk. The last thing I will justify is why we can reduce to the case of L being a finite field.

Proposition 3.4. *Any decent isocrystal N over L is base changed from a finite field.*

Proof. It suffices to consider the case where N has a single slope λ and is generated by elements n with $\mathbf{F}^s n = p^r n$ for some *fixed* $r, s > 0$, so we assume this. Let V be a $\mathbb{Q}_p$-rational subspace of N such that $N = V \otimes_{\mathbb{Q}_p} K_0$. Let $G = \mathrm{GL}(V)$, so that there is some $b \in B(G)$ for which N is the isocrystal associated to b. Now we claim that since N is decent, b is decent. Indeed, $s\nu(p)$ acts as p^r on N for some suitable s, r with $r/s = \lambda$. It follows that b is decent (with $(b\sigma)^s = s\nu(p)\sigma^s$).

Since b is decent, it is defined over $W(\mathbb{F}_{p^s})[\frac{1}{p}]$, and so N is defined over $L \cap \mathbb{F}_{p^s}$. □

This is convenient, because over a finite field, an isocrystal N is decent iff $\mathbf{F}^s = p^r$ on each slope component N_λ for some sufficiently large $s, r > 0$ with $r/s = \lambda$. Indeed, just pick s large enough so that σ^s fixes the ground field.