FORMULATION OF RZ DATA

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August 2, 2021

Throughout, we will have the following notation:

- L will denote an algebraically closed field of characteristic p.
- $W = W(L)$ and $K_0 = W[\frac{1}{p}]$, with Frobenius σ.
- K will denote a finite extension of K_0.

1. Recall

First we recall the concept of an admissible pair. Let G be a reductive group over $\mathbb{Q}_p$ and V a representation. For $b \in G(K_0)$, recall the functor

$$\mathrm{Rep}_{\mathbb{Q}_p}(G) \to \mathrm{Iso}_{K_0}, \quad V \mapsto (V \otimes K_0, b(\mathrm{id} \otimes \sigma))$$

which only depends on the σ-conjugacy class $[b] \in B(G)$ of b.

If we further have a co-character $\mu\colon \mathbb{G}_{m,K} \to G_K$, the pair (μ, b) induces

$$I\colon \mathrm{Rep}_{\mathbb{Q}_p}(G) \to \mathrm{FilIso}_{K/K_0}, \quad V \mapsto (V_{K_0}, b\sigma, \mathrm{Fil})$$

where the filtration is given by the weight spaces of μ. We called the pair (μ, b) *admissible* if all the associated filtered isocrystals above are admissible (it suffices to check on one faithful representation).

Remark 1.1. Admissibility is a pretty strong condition. For each μ, there are only finitely many choices of b with (μ, b) admissible. By taking V to be one-dimensional representations, it implies, for example, that for any character χ of G defined over $\mathbb{Q}_p$ we must have

$$\langle \mu, \chi \rangle = \mathrm{ord}_p \chi(b).$$

2. Algebraic groups of EL/PEL type

We choose:

- F a finite étale extension of $\mathbb{Q}_p$ (i.e. a product of finite extensions).
- B a finite semisimple central algebra over F.
- V a finite dimensional B-module.

Definition 2.1. Given (F, B, V), the associated algebraic group of EL type is $G = \mathrm{GL}_B(V)$ over $\mathbb{Q}_p$, that is,

$$G(R) = \mathrm{GL}_B(V \otimes_{\mathbb{Q}_p} R) \quad \text{for a } \mathbb{Q}_p\text{-algebra } R.$$

In the PEL case, we require $p \neq 2$ and we also choose:

- $(\cdot, \cdot) \colon V \times V \to \mathbb{Q}_p$ a nondegenerate alternating form.
- $(-)^* \colon B \to B$ an involution such that $(bv, w) = (v, b^* w)$.

Definition 2.2. Given $(F, B, V, (\cdot, \cdot), *)$, the associated algebraic group of PEL type is the similitude group of the data:

$$G(R) = \{g \in \mathrm{GL}_B(V \otimes_{\mathbb{Q}_p} R) \colon \exists c(g) \in R^\times \text{ s.t. } (gv, gw) = c(g)(v, w)\} \quad \text{for a } \mathbb{Q}_p\text{-algebra } R.$$

Example 2.3. We have the following examples.

(a) If $B = F$ and $V = F^d$, then $G = \mathrm{GL}_d$.

(b) If B is a division algebra D and $V = D$, then $G(R) = (D^{\mathrm{op}} \otimes_{\mathbb{Q}_p} R)^\times$.

(c) If $D = F$ and $V = F^d$, equipped with a symplectic (i.e. alternating), we get $G = \mathrm{Res}_{F/\mathbb{Q}_p} GSp(V)$ where $GSp(V)_{/F}$ is the symplectic similitude group.

(d) Let B/F is a quadratic extension and V be a Hermitian space with pairing $\langle \cdot, \cdot \rangle$ over B. We can modify the pairing to

$$(x, y) := \mathrm{Tr}_{B/\mathbb{Q}_p}(i \cdot \langle x, y \rangle)$$

where $i \in B$ satisfy $\bar{i} = -i$. Then $(\cdot, \cdot)$ is alternating, and we have $G = \mathrm{Res}_{F/\mathbb{Q}_p} GU(V)$ where $GU(V)_{/F}$ is the unitary similitude group of V.

3. Simple RZ datum

The input data for the RZ spaces will be a situation as above for either the EL or PEL type, together with a certain admissible pair (μ, b), as follows.

Definition 3.1. A *simple RZ data* of EL/PEL type is a choice of data as above, together with:

- An admissible pair (μ, b) such that:
  (a) The isocrystal $N := (V_{K_0}, b\sigma)$ has slopes in $[0,1]$.
  (b) The weight decomposition of V_K with respect to μ only contains the two weights 0 and 1
$$V_K = V_0 \oplus V_1.$$
  (c) (PEL case) The composition $\mathbb{G}_{m,K} \xrightarrow{\mu} G_K \xrightarrow{c} \mathbb{G}_{m,K}$ is the identity.
- A maximal order $\mathcal{O}_B$ of B, and a $\mathcal{O}_B$-stable lattice Λ of V such that:
  (d) (PEL case) $\mathcal{O}_B$ is stable under $*$.
  (e) (PEL case) $\Pi\Lambda^\vee \subseteq \Lambda \subseteq \Lambda^\vee$ for a prime Π of $\mathcal{O}_B$.[1]

Secretly, we are thinking of this data as coming from a p-divisible group X over $\mathcal{O}_K$ with $\mathcal{O}_B$ action: the isocrystal N is the isocrystal of the reduction to $\mathcal{O}_L$, and the weight decomposition is the canonical filtration

$$0 \to \mathrm{Fil}^1 \to M(X) \otimes \mathbb{Q} \to \mathrm{Lie}(X) \otimes \mathbb{Q} \to 0.$$

In the PEL case, we think of X as also having a polarization, that is, a map $X \to X^\vee$ that is anti-symmetric for the $\mathcal{O}_B$ action.

We denote by $J(\mathbb{Q}_p)$ the group of endomorphisms of the isocrystal N.

Remark 3.2. Some notes about the isocrystal N:

- It has an action of B.
- In the PEL case, we have an alternating form induced from the one on V:
$$\psi \colon N \times N \to (K_0, c(b)\sigma) \xrightarrow{\sim} K_0(n),$$
where $n = \mathrm{ord}_p c(b)$. In fact, condition Definition 3.1(c) together with Remark 1.1 implies that $n = 1$, since $\langle \mu, c \rangle = 1$.

Remark 3.3. The condition Definition 3.1(a) means that the isocrystal $(V_{K_0}, b\sigma)$ is associated to a p-divisible group $\mathbb{X}$ over L. In the PEL case, the $\mathbb{Q}_p^\times \psi$ above induces an anti-symmetric quasi-isogeny
$$\lambda \colon \mathbb{X} \to \mathbb{X}^\vee,$$

[1]Here, $\Lambda^\vee = \{v \in V : (v, \Lambda) \subseteq \mathbb{Z}_p\}$. In the unitary case Example 2.3(d), $\Lambda^\vee$ may not be the same as the Hermitian dual if the data is ramified.

and $\mathbb{Q}_p^\times \lambda$ is determined from the data above. Note that the Rosati involution induces on $\mathcal{O}_B$ the involution $*$.

Example 3.4. We have the following examples of simple RZ datum.

(1) (Lubin–Tate case) This is EL type, with $B = \mathbb{Q}_p$ and $V = \mathbb{Q}_p^d$, so $G = \mathrm{GL}_d$. Here $\mu(t) = \mathrm{diag}(t, \ldots, t, 1)$, and

$$b = \begin{pmatrix} 0 & & & 1 \\ p & \ddots & & \\ & \ddots & \ddots & \\ & & p & 0 \end{pmatrix} \in \mathrm{GL}_d(K_0).$$

Then the isocrystal N is K_0^d with $\Phi = b\sigma$. Note that on $\mathbb{Z}_p^d$, we have $\Phi^d = p^{d-1}$, and so N has slope $\frac{d-1}{d}$. It is associated with the Lubin–Tate formal groups, and $J(\mathbb{Q}_p) = D_{1/d}^\times$ where $D_{1/d}$ is the division algebra over $\mathbb{Q}_p$ with invariant $1/d$.

(2) (Drinfeld case) This is a EL type, with $B = D$ a central division algebra over $\mathbb{Q}_p$ of invariant $1/d$, and $V = D$. Concretely, $D = F_d\{\pi\}/(\pi^d - p, \pi x - x^\sigma \pi)$ for $F_d/\mathbb{Q}_p$ unramified of degree d and $\mathcal{O}_D = \mathcal{O}_{F_d}\{\pi\}/(\pi^d - p, \pi x - x^\sigma \pi)$. Now

$$G(K_0) = (D^{\mathrm{op}} \otimes_\rho K_0)^\times$$

and we consider its element $b = \pi^{d-1}$. Then it is clear all slopes are $\frac{d-1}{d}$. μ is such that V_0 is a d-dimensional D stable subspace. In this case, we have $J(\mathbb{Q}_p) = \mathrm{GL}_d$.

(3) This is PEL type, with $B = E$ a quadratic extension of F. Let $\tau \colon E \to K$ be a prefered embedding. Let V be a Hermitian E-space, and choose a $\mathcal{O}_E$-lattice Λ such that $\varpi\Lambda^\vee \subseteq \Lambda \subseteq \Lambda^\vee$. Choose a signature (r, s) with $r + s = \dim_E V$. Then there is a choice of data (μ, b) such that V_0 is a subspace where the action of E is diagonalized to $\tau(a)^{\oplus r} \oplus \tau^c(a)^{\oplus s}$. This is associated to the p-divisible group of an abelian scheme with action by $\mathcal{O}_E$ and CM type (r, s).

4. RZ spaces for simple data

Let E be the reflex field of the conjugacy class of μ, and let $\breve{E}$ be the completed unramified extension of E, with residue field L. Note that if $S \to \operatorname{Spec} \mathcal{O}_{\breve{E}}$, then $\overline{S} \to \operatorname{Spec} \mathcal{O}_{\breve{E}}/p\mathcal{O}_{\breve{E}} \to \operatorname{Spec} L$.

Definition 4.1. Let $\mathscr{D}$ be a simple RZ data of EL resp. PEL type. We define the functor $\breve{\mathscr{M}}\colon \mathrm{Nil}^{\mathrm{op}}_{\mathcal{O}_{\breve{E}}} \to \mathrm{Set}$ with values in $S \in \mathrm{Nil}_{\mathcal{O}_{\breve{E}}}$ to be the set of tuples up to isomorphism (X, ι, ρ) resp. $(X, \iota, \lambda, \rho)$ where:

- X is a p-divisible group over S.
- ι is an action of $\mathcal{O}_B$ on X, that is $\iota\colon \mathcal{O}_B \to \mathrm{End}(X)$.
- $\rho\colon \mathbb{X}_{\overline{S}} \to X_{\overline{S}}$ is a quasi-isogeny which commutes with the action of $\mathcal{O}_B$.
- (PEL case) $\lambda\colon X \to X^\vee$ is an isogeny.

We require that this data satisfies:

(a) If $M(X)$ if the Dieudonné module of X, which is a $\mathcal{O}_B \otimes \mathcal{O}_S$ module, then $M(X)$ is locally on S isomorphic to $\Lambda \otimes \mathcal{O}_S$.

(b) (Kottwitz condition) We have an equality of polynomials

$$\mathrm{char}_{\mathcal{O}_S}(\iota(a);\ \mathrm{Lie}(X)) = \mathrm{char}_K(a; V_0) \quad \text{for all } a \in \mathcal{O}_B.$$

Here the left side has coefficients in $\mathcal{O}_S$ and the right hand side in $\mathcal{O}_E$, and we compare them via the structure morphism. In the PEL case, we also assume this for $X^\vee$.[2]

(c) (PEL case) The isogeny $\lambda\colon X \to X^\vee$ fits in the following diagram which commutes up to a constant in $\mathbb{Q}_p^\times$,

$$\begin{array}{ccc} \mathbb{X}_{\overline{S}} & \xrightarrow{\lambda} & \hat{\mathbb{X}}_{\overline{S}} \\ \big\downarrow{\scriptstyle\rho} & & \big\uparrow{\scriptstyle\hat{\rho}} \\ X_{\overline{S}} & \xrightarrow{\lambda} & X^\vee_{\overline{S}} \end{array}$$

and is such that the induced inclusion $M(X) \to M(X^\vee)$ has cokernel locally on S isomorphic to $(\Lambda^\vee/\Lambda) \otimes \mathcal{O}_S$.

Remark 4.2. The functor above is independent of the choice of $\mathbb{X}$: it only depends on the isocrystal N and its extra structure. Hence, the automorphism group $J(\mathbb{Q}_p)$ of N together with its polarization acts on $\breve{\mathscr{M}}$.

Remark 4.3. A few remarks about condition Definition 3.1(b): Write $B = \prod_{i=1}^n M_{n_i}(D_i)$ for division algebras D_i, and choose this identification such that $\mathcal{O}_B = \prod_{i=1}^n M_{n_i}(\mathcal{O}_{D_i})$. This induces $\Lambda = \bigoplus_{i=1}^n \Lambda_i$ and $X = \prod_{i=1}^n X_i$. The Kottwitz condition is a condition on each component, so let's assume that $n = 1$. Let $F = Z(D)$ and $\tilde{F}$ an unramified extension contained in D that splits D.

[2]This may be automatic from the following condition in some cases, but it is required in general.

Concretely, we have $D = \tilde{F}\{\Pi\}/(\Pi^d - \pi, \Pi x - x^\tau \Pi)$ for $\tau = (\mathrm{Frob}_{\tilde{F}/F})^s$ where $(d,s) = 1$. Let F^t, resp $\tilde{F}^t$ the maximal unramified subextensions, and choose K such that $\tilde{F}^t \subseteq K$. Then we have

$$V_0 = \bigoplus_{\phi\colon \tilde{F}^t \to K} V_0^\phi.$$

Note that conjugation by Π induces τ, and so multiplication by Π induces $\Pi\colon V_0^{\phi\tau} \xrightarrow{\sim} V_0^\phi$. So it follows that if $\phi|_{F^t} = \phi'|_{F^t}$, then V_0^ϕ and $V_0^{\phi'}$ have the same rank over K. For $S \in \mathrm{Nil}_{\mathcal{O}_{\breve{E}}}$, we also have

$$\mathcal{O}_{\tilde{F}^t} \otimes \mathcal{O}_S = \prod_{\phi\colon \tilde{F}^t \to K} \mathcal{O}_S$$

and so

$$\mathrm{Lie}(X) = \bigoplus_{\phi\colon \tilde{F}^t \to K} \mathrm{Lie}^\phi(X).$$

And so the restriction of Definition 3.1(b) to the subalgebra $\mathcal{O}_{\tilde{F}^t}$ of $\mathcal{O}_B$ says that the rank of $\mathrm{Lie}^\phi(X)$ as a locally free $\mathcal{O}_S$-module is the same as the dimension of V_0^ϕ over K.

If B is an unramified local field, than this is equivalent to the Kottwitz condition.

Remark 4.4. Condition Definition 3.1(a) follows from the Kottwitz condition Definition 3.1(b). To see this, note that $M(X) = \prod_{i=1}^n M_i$, and the Kottwitz condition is saying that the $M_{n_i}(\mathcal{O}_{D_i}) \otimes \mathcal{O}_S$ is locally free of same rank as Λ_i. By Morita equivalence, it suffices to check this when $B = D$. The rank condition is automatic from the existence of the quasi-isogeny ρ. So it suffices to see that $M(X)$ is a locally free $\mathcal{O}_D \otimes \mathcal{O}_S$-module. We know it is a locally free $\mathcal{O}_S$-module, so it suffices to see that for any geometric point $\operatorname{Spec} P \to S$, we have that $M(X) \otimes_{\mathcal{O}_S} P$ is a free $\mathcal{O}_D \otimes P$-module. Let M_P denote the crystal at $\operatorname{Spec} P$, which is a $\mathcal{O}_D \otimes W(P)$-module. From the decomposition $\mathcal{O}_{\tilde{F}^t} \otimes W(P) = \bigoplus_{\phi\colon \tilde{F}^t \to W(P)[\frac{1}{p}]} W(P)$, we have $M_P = \bigoplus_\phi M^\phi$. As before, we have $\Pi\colon M^{\phi\tau} \to M^\phi$, and denote by C_ϕ the cokernel. So

$$M_P/\Pi M_P \simeq \bigoplus_\phi C_\phi.$$

Choosing representatives for a basis for each C_ϕ and letting $\mathcal{O}_D$ act, we get

$$M_P \simeq \bigoplus_\phi (\mathcal{O}_D \otimes_\phi W(P))^{\dim_P C_\phi}.$$

We will prove that all $\dim_P C_\phi$ are equal, so that M_P is a free $\mathcal{O}_D \otimes W(P)$-module. This follows from the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M^{\mathrm{Frob}\phi\tau} & \xrightarrow{\Pi} & M^{\mathrm{Frob}\phi} & \longrightarrow & C^{\mathrm{Frob}\phi} & \longrightarrow & 0 \\
 & & \downarrow V & & \downarrow V & & \downarrow & & \\
0 & \longrightarrow & M^{\phi\tau} & \xrightarrow{\Pi} & M^{\phi} & \longrightarrow & C^{\phi} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{Lie}^{\phi\tau} & \longrightarrow & \mathrm{Lie}^{\phi} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

since the Kottwitz condition implies that $\mathrm{Lie}^{\phi\tau}$ and Lie^{ϕ} have the same dimension.

Remark 4.5. Similarly, we can see that the last part of Definition 3.1(c) is implied by the Kottwitz condition Definition 3.1(b) together with:

$$(\star) \qquad \text{the height of } \lambda \text{ is } \log_p|\Lambda^\vee/\Lambda|, \text{ and } \ker\lambda \subseteq X[\iota(\varpi)].$$

To see this, note that, as before, it is a statement about local freeness and correct rank. The condition above ($\star$) guarantess the correctness of the rank once we prove the local freeness. As before, we may assume $B = D$. Let C be the cokernel of $M(X) \to M(X^\vee)$. Note that from $\ker\lambda \subseteq X[\iota(\varpi)]$, this cokernel is a $\mathcal{O}_S/p\mathcal{O}_S$-module. It suffices to prove it is a locally free $(\mathcal{O}_D/\Pi)\otimes \mathcal{O}_S/p\mathcal{O}_S$-module. It is a general result on isocrystals that C is a locally free $\mathcal{O}_S/p\mathcal{O}_S$-module, so as before it suffices to consider $S = P$ algebraically closed. As before, it remains to see that N^ϕ have all the same dimension, and this follows from the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M(X)^{\sigma\phi} & \longrightarrow & M(X^\vee)^{\sigma\phi} & \longrightarrow & C^{\sigma\phi} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle V} & & \downarrow{\scriptstyle V} & & \downarrow & & \\
0 & \longrightarrow & M(X)^{\phi} & \longrightarrow & M(X^\vee)^{\phi} & \longrightarrow & C^{\phi} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{Lie}(X)^{\phi} & \longrightarrow & \mathrm{Lie}(X^\vee)^{\phi} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

and the fact that both X and $X^\vee$ satisfy the Kottwitz condition.

With the above remarks, we may rephrase the definition of the moduli functor as follows:

Definition 4.6 (Revised Definition)**.** Let $\mathscr{D}$ be a simple RZ data of EL resp. PEL type. We define the functor $\breve{\mathscr{M}} \colon \mathrm{Nil}^{\mathrm{op}}_{\mathcal{O}_{\breve{E}}} \to \mathrm{Set}$ with values in $S \in \mathrm{Nil}_{\mathcal{O}_{\breve{E}}}$ to be the set of tuples up to isomorphism (X, ι, ρ) resp. $(X, \iota, \lambda, \rho)$ where:

- X is a p-divisible group over S.
- ι is an action of $\mathcal{O}_B$ on X, that is $\iota \colon \mathcal{O}_B \to \mathrm{End}(X)$.
- $\rho \colon \mathbb{X}_{\overline{S}} \to X_{\overline{S}}$ is a quasi-isogeny which commutes with the action of $\mathcal{O}_B$.
- (PEL case) $\lambda \colon X \to X^\vee$ is an isogeny.

We require that this data satisfies:

(a) (Kottwitz condition) We have an equality of polynomials

$$\mathrm{char}_{\mathcal{O}_S}(\iota(a);\ \mathrm{Lie}(X)) = \mathrm{char}_K(a; V_0) \quad \text{for all } a \in \mathcal{O}_B,$$

In the PEL case, we also assume this for $X^\vee$.

(b) (PEL case) The isogeny $\lambda \colon X \to X^\vee$ has height $\log_p|\Lambda/\Lambda^\vee|$, satisfies that $\ker\lambda \subseteq X[\iota(\varpi)]$, and fits in the following diagram which commutes up to a constant in $\mathbb{Q}_p^\times$.

$$\begin{array}{ccc}
\mathbb{X}_{\overline{S}} & \xrightarrow{\lambda} & \hat{\mathbb{X}}_{\overline{S}} \\
\downarrow{\scriptstyle \rho} & & \uparrow{\scriptstyle \hat{\rho}} \\
X_{\overline{S}} & \xrightarrow{\lambda} & X^\vee_{\overline{S}}
\end{array}$$

5. General RZ datum and spaces

The general version of RZ data replaces the lattices Λ by a multichain of lattices, as we will define.

Definition 5.1. Let B be a dinite dimensional simple algebra over $\mathbb{Q}_p$. A chain of lattices $\mathcal{L}$ is a set of $\mathcal{O}_B$-lattices of V such that:

- If $\Lambda, \Lambda' \in \mathcal{L}$, then either $\Lambda \subseteq \Lambda'$ or $\Lambda' \subseteq \Lambda$.
- If $x \in B^\times$ normalizes $\mathcal{O}_B$, then $x\Lambda \in \mathcal{L} \iff \Lambda \in \mathcal{L}$.

More explicitly, if $B = M_n(D)$ for a division algebra D, with $\mathcal{O}_B = M_n(\mathcal{O}_D)$, then the normalizer of $\mathcal{O}_B$ is $D^\times \mathcal{O}_B^\times$. So if Π is a prime of $\mathcal{O}_D$, the second condition is equivalent to

$$\Lambda \in \mathcal{L} \iff \Pi\Lambda \in \mathcal{L}.$$

Hence, giving a chain $\mathcal{L}$ is equivalent to giving a collection of lattices

$$\Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_{r-1} \subset \frac{1}{\Pi}\Lambda_0.$$

Let $B = \prod_{i=1}^m B_i$ be a finite dimensional semisimple algebra over $\mathbb{Q}_p$, with decomposition such that $\mathcal{O}_B = \prod_{i=1}^m \mathcal{O}_{B_i}$. Then we get a corresponding decomposition $V = \bigoplus_{i=1}^m V_i$, and any lattice Λ is uniquely written as $\Lambda = \bigoplus_{i=1}^m \Lambda_i$.

Definition 5.2. A multichain of $\mathcal{O}_B$-lattices $\mathcal{L}$ in V is a collection such that there are chains of lattices $\mathcal{L}_i$ such that $\mathcal{L} = \{\Lambda \colon \Lambda_i \in \mathcal{L}_i \text{ for all } i\}$.

In the PEL case, the pairing and the involution allow us to take dual lattices: for a lattice Λ, we let $\Lambda^\vee = \{v \in V \colon (v, \Lambda) \subseteq \mathbb{Z}_p\}$.

Definition 5.3. A multichain $\mathcal{L}$ of lattices is self dual if $\Lambda \in \mathcal{L} \iff \Lambda^\vee \in \mathcal{L}$.

Definition 5.4. A *RZ data* of EL/PEL type is a choice of (F, B, V, μ, b) as before, and $((\cdot,\cdot), *)$ in the PEL case, together with a maximal order $\mathcal{O}_B$ of B and a multichain of lattices $\mathcal{L}$, which is self-dual in the PEL case.

With this, we define the general RZ spaces.

Definition 5.5. Let $\mathscr{D}$ be a RZ data of EL/PEL type. We define the functor $\breve{\mathscr{M}} \colon \mathrm{Nil}^{\mathrm{op}}_{\mathcal{O}_{\breve{E}}} \to \mathrm{Set}$ with values in $S \in \mathrm{Nil}_{\mathcal{O}_{\breve{E}}}$ to be the set of tuples $(X_\Lambda, \iota_\Lambda, \rho_\Lambda)_{\Lambda \in \mathcal{L}}$ up to isomorphism where:

- X_Λ is a p-divisible group over S.
- ι_Λ is an action of $\mathcal{O}_B$ on X, that is, $\iota_\Lambda \colon \mathcal{O}_B \to \mathrm{End}(X_\Lambda)$.
- $\rho_\Lambda \colon \mathbb{X}_{\overline{S}} \to X_{\Lambda,\overline{S}}$ is a quasi-isogeny which commutes with the action of $\mathcal{O}_B$.

Denote $\tilde{\rho}_{\Lambda',\Lambda} \colon X_\Lambda \to X_{\Lambda'}$ the quasi-isogeny lifting $\rho_{\Lambda'}\rho_\Lambda^{-1}$. We require that this data satisfies:

(a) If $M(X_\Lambda)$ if the Dieudonné module of X, which is a $\mathcal{O}_B \otimes \mathcal{O}_S$ module, then $M(X_\Lambda)$ is locally on S isomorphic to $\Lambda \otimes \mathcal{O}_S$.

(b) (Kottwitz condition) We have an equality of polynomials

$$\mathrm{char}_{\mathcal{O}_S}(\iota(a);\ \mathrm{Lie}(X_\Lambda)) = \mathrm{char}_K(a; V_0) \quad \text{for all } a \in \mathcal{O}_B.$$

(c) If $\Lambda \subseteq \Lambda'$, then $\tilde{\rho}_{\Lambda',\Lambda}$ is an isogeny, and the cokernel of the induced map $M(X_\Lambda) \to M(X_{\Lambda'})$ is locally on S isomorphic to $\Lambda'/\Lambda \otimes \mathcal{O}_S$ as a $\mathcal{O}_B \otimes \mathcal{O}_S$-module.

(d) For any $a \in B^\times$ normalizing $\mathcal{O}_B$, if we denote X_Λ^a to be the pair (X_Λ, ι^a) where $\iota^a(x) = \iota_\Lambda(a^{-1}xa)$, then multiplication by $\iota_\Lambda(a)$ induces an isomorphism

$$X^a \xrightarrow{\sim} X_{a\Lambda}.$$

(e) (PEL case) For each $\Lambda \in \mathcal{L}$, there is an isomorphism $p_\Lambda \colon X_\Lambda \to X_{\Lambda^\vee}^\vee$ which fits in the following diagram, which commutes up to a constant in $\mathbb{Q}_p^\times$ independent of Λ.

$$\begin{array}{ccc} \mathbb{X}_{\overline{S}} & \xrightarrow{\lambda} & \hat{\mathbb{X}}_{\overline{S}} \\ \downarrow{\scriptstyle \rho_\Lambda} & & \uparrow{\scriptstyle \hat{\rho}_{\Lambda^\vee}} \\ X_{\Lambda,\overline{S}} & \xrightarrow{p_\Lambda} & X^\vee_{\Lambda^\vee,\overline{S}} \end{array}$$

Remark 5.6. As before, condition Definition 5.5(a) can be removed, and Definition 5.5(c) can be replaced by

(c') If $\Lambda \subseteq \Lambda'$, then $\tilde{\rho}_{\Lambda',\Lambda}$ is an isogeny of height $\log_p|\Lambda'/\Lambda|$.

6. Representability

Our Kottwitz condition is slightly different from the original condition defined in Rapoport–Zink. They are equivalent.[3] Our condition is easier to describe, but their condition is more clearly a closed condition.

[3]See Proposition 2.1.3 of arXiv:1210.1559 for a proof.

Their condition is as follows. We only compare $\det_{\mathcal{O}_S}(a; \mathrm{Lie}(X)) = \det_K(a; V_0)$, but we have to do this as polynomial functions on a: Let $\mathbb{V}_{/\mathbb{Z}_p}$ be the scheme where $\mathbb{V}(R) = \mathcal{O}_B \otimes_{\mathbb{Z}_p} R$. Choose $\Gamma \subseteq V_0$ an $\mathcal{O}_B$-invariant lattice. We define $\mathbb{V}_{\mathcal{O}_K} \to \mathbb{A}^1_{\mathcal{O}_K}$ by $a \in \mathcal{O}_B \otimes_{\mathcal{O}_K} R \mapsto \det(a; \Gamma \otimes_{\mathcal{O}_K} R)$. This morphism is defined over $\mathcal{O}_E$, and does not depend on the choice of Γ. Similarly, we can make $\det(a; \mathrm{Lie}(X))$ as a morphism $\mathbb{V}_S \to \mathbb{A}^1_S$. We then require that these morphisms agree over S.

Note that the agreement of these morphisms, and hence the Kottwitz condition, is a closed condition. With this, we can prove the representability of $\breve{\mathscr{M}}$.

Theorem 6.1. *The functor $\breve{\mathscr{M}}$ is representable by a formal scheme which is formally locally of finite type over* $\operatorname{Spf} \mathcal{O}_{\breve{E}}$.

Proof. Let $\mathscr{M}$ be the representable functor from last week for $\mathbb{X}$ (since L is algebraically closed, $\mathbb{X}$ is automatically decent). We may transport the action of $\mathcal{O}_B$ on $\mathbb{X}$ to an action of $\mathcal{O}_B$ on the points of $\mathscr{M}$ by quasi-isogenies. Then the subfunctor $\mathscr{M}_{\mathcal{O}}$ for where these quasi-isogenies are actually isogenies is a closed subfunctor. Now we can consider the obvious morphism

$$j \colon \breve{\mathscr{M}} \to \prod_{\Lambda \in \mathcal{L}} \mathscr{M}_{\mathcal{O}},$$

and we will prove that this is a closed immersion. We will refer to the conditions of Definition 5.5. The condition (b) is representable by the discussion above, and by the Remark 5.6 this implies (a). Conditions (d) and (e) are closed conditions. For (c), it suffices to check Remark 5.6(c'): $\tilde{\rho}_{\Lambda',\Lambda}$ being an isogeny is a closed condition, while specifying the height is an open and closed condition. Hence j is a closed immersion, and this implies $\breve{\mathscr{M}}$ is representable.

In fact, the morphism j factors through

$$\breve{\mathscr{M}} \to \prod_{\Lambda} \mathscr{M}_{\mathcal{O}}$$

for any finite collection of Λ's which generate $\mathcal{L}$ via multiplication by elements that normalize $\mathcal{O}_B$. Since $\mathscr{M}$ is formally locally of finite type, this implies $\breve{\mathscr{M}}$ is as well. □