Formulation of moduli functors of quasi-isogenies, representability

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Today, I will focus on the remaining part of Chapter 2 of [1].

We begin by recall our moduli problem

1 Statement of Moduli Problem

Theorem 1.1. Let $\mathbb{X}$ be a decent p-divisible group over a perfect field L, $W = W(L)$, consider the moduli functor $\mathcal{M} : \mathrm{Nilp}_W \to \mathrm{Sets}$, where

$$\mathcal{M}(S) = \{(X, \rho) | \rho : \mathbb{X} \times_L \bar{S} \dashrightarrow X \times_S \bar{S}\} / \cong$$

Where $(X_1, \rho_1) \cong (X_2, \rho_2)$ if and only if $\rho_1 \circ \rho_2^{-1}$ lifts to an isomorphism $X_2 \to X_1$.

Then $\mathcal{M}$ is representable by a formal scheme over $\mathrm{Spf}\, W$, which is formally of locally of finite type, and each irreducible component of $\mathcal{M}_{\mathrm{red}}$ is projective over L.

Remark.

- Let $J(\mathbb{Q}_p)$ be the group of quasi-isogeny on X, acts on the right on $\mathcal{M}$, by $(X, \rho) \cdot \gamma = (X, \rho \circ \gamma)$. Hence acts on geometric spaces and their cohomology groups constructed from $\mathcal{M}$.
- Drinfeld rigidity lemma implies $\mathrm{Aut}(X, \rho) = \{\mathrm{id}\}$
- The moduli functor $\mathcal{M} = \mathcal{M}_{\mathbb{X}}$ only depends on isogeny class of $\mathbb{X}$.

In order to understand the "discrete part" of $\mathcal{M}$, we introduce

Definition 1.2.

(1) If $f : X \to Y$ is an isogeny of p-divisible groups over S. Then the order of $\ker f$ is p^h, for some $\mathbb{Z}_{\geq} 0$ valued, locally constant function h. If h is constant, we call it the **height** of f.

(2) $f : X \dashrightarrow Y$ be a quasi-isogeny. Assume $p^n f$ is an isogeny, we define the **height** of f by

$$\mathrm{ht}(f) = \mathrm{ht}(p^n f) - \mathrm{ht}(p^n)$$

For example, the height of multiplication by p is the height of X.

Remark. For isogenies, one has $\mathrm{ht}(f_1 \circ f_2) = \mathrm{ht}(f_1) + \mathrm{ht}(f_2)$. Thus the height of a quasi-isogeny is well-defined, and the above relation also holds for quasi-isogenies.

Height is a discrete invariant of a quasi isogeny. Define $\mathcal{M}(h)(S) = \{(X,\rho) | \operatorname{ht}(\rho) = h\}$. Then $\mathcal{M}(h)$ is an open and closed functor of $\mathcal{M}(h)$ is an open and closed subfunctor of $\mathcal{M}$. And $\mathcal{M} = \bigsqcup \mathcal{M}(h)$. Thus, it suffices to show each $\mathcal{M}(h)$ is representatble. Or one can define $\widetilde{\mathcal{M}} = \bigsqcup_{h=0}^{\operatorname{ht}\mathbb{X}-1} \mathcal{M}(h)$, and it suffices to show $\widetilde{\mathcal{M}}$ is representable.

2 Some examples

We focus on $L = \bar{L}$ and $\mathbb{X}$ is a height 2, dim 1 p divisible group. We study the correponding functor.

Up to isogeny, they are classified by associated isocrystal, by Dieudonné-Manin classification, they are classified by Newton polygons from $(0,0)$ to $(2,1)$.

Remark. We have the following facts: for a p-divisible group G over a perfect field k of characteristic p

- G is étale $\iff D(G)_{\mathbb{Q}}$ is isoclinic of slope 0
- G is formal(defined later) $\iff D(G)_{\mathbb{Q}}$ has no zero slope.

Now we consider associated $\mathcal{M}$ for these $\mathbb{X}$.

Example 2.1. When $\mathbb{X} = E[p^\infty]$, where E supersingular ellipitic curve over L. We will show that as a formal scheme

$$\mathcal{M} = \bigsqcup_{h\in\mathbb{Z}} \operatorname{Spf}(W[[x]])$$

Remark. $\mathcal{M}_{\text{red}} = \mathcal{M}(L)$ is disjoint union of points

We will show a more general result:

Proposition 2.2. If $\mathbb{X}$ comes from a formal group of dim 1, height n over $L = \bar{L}$, then

$$\mathcal{M}_{\mathbb{X}} = \bigsqcup_{h\in\mathbb{Z}} \operatorname{Spf}(W(L)[[x_1,\cdots,x_n]]$$

Some background:

Definition 2.3. An n **-dimensional commutative formal group law** over ring A = a power series $F \in A[[X_1,\cdots,X_n,Y_1,\cdots,Y_n]]$ which satisfies some formal group axioms.

For example, we have $\widehat{\mathbb{G}_a} = X+Y, \widehat{\mathbb{G}_m} = X+Y+XY$, or completion of abelian scheme over A along zero section, or Lubin-Tate formal group appeared in local class field theory.

Definition 2.4. We say F is p**-divisible** if $[p] : A[[X]] \to A[[X]]$ is finite locally free. The rank of $[p]$ is p^h for some h. h is called **height**.

One has the following result of Tate and Messing

Theorem 2.5 (Tate-Messing)**.** The category of p-divisible formal group fully faithfully embeds in to category of p-divisible groups, which preserve height and dimension.

The essential image above is called **formal**p-divisible groups.

We will mainly focus on 1-dim formal group law. When F is a 1-dim formal group law over a field of characteristic p, the notions above can be characterized in an easier way:

Proposition 2.6. For 1-dim formal group law F over a field of characteristic p, then F is p-divisible if and only if $[p] \neq 0$. In this case, $[p] = g(X^{p^h})$ for some g with $g'(0) \neq 0$. Such h is the height of F.

We $[p] = 0$, we also say F has height ∞.

Theorem 2.7. If $L = \bar{L}$, then for each height $h \in \{1, 2, \cdots \infty\}$, there exists a unique (up to isomorphism) 1-dim formal group F_0 of height h over L.

Fact: $\mathrm{End}(F) = \mathbb{Z}_{p^h}[\Pi], \Pi^h = p, \Pi a = \sigma(a)\Pi$.

We can consider the following Lubin-Tate deformation functor: Fix F above. Define $\mathcal{C}_L$, the category of local Artin rings with a fixed surjection $A \twoheadrightarrow L$

Consider the following deformation functor:

$$\mathcal{D} : \mathcal{C}_L \to \mathrm{Sets}$$

which sends $A \twoheadrightarrow L$ to isomorphism classes of $\{F, \iota\}$, where F is a formal group law over A and ι is an isomorphism $F \otimes_A L \cong F_0$.

One has the following theorem of Lubin and Tate

Theorem 2.8. $\mathcal{D}$ is representable by $\mathrm{Spf}(W(k)[[x_1, \cdots, x_{n-1}]]$

Now we find its relationship with R-Z moduli problem

Lemma 2.9. For 1-dim p divisible formal group, quasi-isogeny of height 0 is equivalent to an isomorphism

Proof. Self quasi-isogeny is $D^\times$, and both morphisms above corresponds to $\mathcal{O}_D^\times$. □

Now if R is artinian local ring, $\mathrm{Spec}\, R \in \mathrm{Nil}_W$.

Proposition 2.10. $\mathcal{D}(R) \cong \mathcal{M}(0)(R)$ canonically.

Proof. $\mathcal{M}(0)(R)$ consists of quasi-isogeny of height 0 on R/p, which is equivalent to quasi-isogeny of height 0 on L, by Drinfeld rigidity, thus by the lemma, which is equivalent to an isomorphism. And by a result of Tate, X is p-divisible over $R \in \mathcal{C}_L$, then X is connected if and only if X is formal. Thus, $\mathcal{M}(0)(R)$ is equivalent to $\mathcal{D}(R)$ □

Remark.
- We didn't prove $\mathcal{M}(0)$ is representable by $\sqcup \mathrm{Spf}\, W(k)[[x_1, \cdots, x_{n-1}]]$ above. (Since we need to compare to other $S \in \mathrm{Nil}_W \backslash \mathcal{C}_L$. But once we know $\mathcal{M}$ representable and is formally of locally of finite type, and $\mathcal{M}$ is a point, then we can conclude $\mathcal{M}(0) = \mathcal{D}$.
- Since Π is an isogeny of height 1, thus for each height, we have isogeny of height 1, thus $\mathcal{M}$ is disjoint union of $\mathrm{Spf}\, W(k)[[x_1, \cdots, x_{n-1}]]$, parametrized by height.

Example 2.11. Let's briefly talk about another example, where $\mathbb{X} = E[p^\infty]$, for an ordinary elliptic curve E. In this case, $\mathcal{M} = \bigsqcup_{\mathbb{Z}^2} \widehat{\mathbb{G}_m}$, which is a restatement of theory of Serre-Tate coordinates.

3 Proof strategy

Recall that, in the previous week, we have reduced to the case where L is a finite field, and $\mathbb{X}$ lifts to $\widetilde{\mathbb{X}}$ on $\operatorname{Spf} W(L)$.

The proof strategy is

(1) Approximate $\mathcal{M}$ by "controllable" subfunctors $\mathcal{M}_n$ of $\mathcal{M}$

(2) Show that each $\mathcal{M}_n$ is representable by a formal scheme, and underlying reduced schemes are eventually the same.

(3) Take $\mathcal{M}$ as same space + "limit of sheaves", then show it is a formal scheme, which really represents $\mathcal{M}$.

Here are some definitions related to "controllable" ones,

Definition 3.1. (1) Define $\mathcal{M}^n(S) = \{(X, \rho) \in \mathcal{M}(S) | p^n \rho \text{is an isogeny}\}$.

(2) For quasi-isogeny $\alpha : X \dashrightarrow Y$, define $q(\alpha) = \mathrm{ht}(p^n \alpha)$, where n is smallest integer such that $p^n \alpha$ is an isogeny,

(3) Define $d(\alpha) = q(\alpha) + q(\alpha^{-1})$.

(4) Define $\mathcal{M}_c(S) = \{(X.\rho) \in \mathcal{M}(S) | d(\rho_s) \leq c \text{for any } s \in S\}$.

d is like kind of a metric:

Lemma 3.2. $d(\alpha) + d(\beta) \geq d(\alpha + \beta)$

Proof. Easy, reduced to additivity of height function. □

Now we can state our strategy more clearly:

(1) Each $\mathcal{M}_n$ is representable.

(2) $\mathcal{M}_c^n$ is representable.

(3) Underlying reduced scheme of $\mathcal{M}_c^n$ are eventually the same

(4) $\mathcal{M}_c$ is representable.

(5) $\mathcal{M}$ is representable.

4 Some details of the proof

Proposition 4.1. $\mathcal{M}^n$ is representable

Proof. Define $\mathcal{M}^{n,m} = \{(X, \rho) | p^n \rho \text{is an isogeny of height} m\}$, which is an open and closed subfunctor of $\mathcal{M}$. Suffices to show each $\mathcal{M}^{n,m}$ is representable.

But give such ρ is equivalent to give an isogeny of height m, which corresponds to locally free subgroup of $\widetilde{X}[m]_S$ of order p^n. Thus it is representable by a closed subscheme of Grassmanian (consists of Hopf ideals, which is defined by a polynomial relation). And, we can also take p-adic completion of this scheme, since p is locally nilpotent, thus after taking completion, it represents the same functor. □

Proposition 4.2. $\mathcal{M}_c^n$ is representable.

Lemma 4.3. For $\alpha : X \to Y$ an isogeny, $\{s \in S | d(\alpha_s) \leq c\}$ is closed.

Proof. Not hard, it follows from the subsets of S where a given quasi-isogeny is an isogeny, which is a proposition proved last time. □

Proof of 4.2:Consider the universal p-divisible group on $\mathcal{M}^n$, since the sets of points where $d(\alpha_s) \leq c$ is closed, taking completion of it suffices.

The last three steps are more complicated, which is based on the key lemma Prop 2.17 in [1].

References

[1] Rapoport-Zink. *Period Spaces of p-divisible Groups.*