Local models for Rapoport-Zink spaces

Seminar notes

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1 Overview

We discuss *local models* $\mathbb{M}^{\mathrm{loc}}$ for the Rapoport-Zink spaces $\breve{\mathcal{M}}$ of [RZ96, §3], discussed in Murilo's talk. Each local model $\mathbb{M}^{\mathrm{loc}}$ will be a closed subscheme of a product of Grassmanians over $\mathcal{O}_E$ (notation reviewed in Section 2), defined by "linear algebraic" conditions suitable for explicit computation. We give several example computations of local models in Section 3.2.

Following [RZ96, §3.26-3.35], we will see in Section 4 that the structure of $\breve{\mathcal{M}}$ is controlled, formally étale locally, by its associated local model $\mathbb{M}^{\mathrm{loc}}$. More precisely, let $\hat{\mathbb{M}}^{\mathrm{loc}}$ denote the p-adic completion of $\mathbb{M}^{\mathrm{loc}} \times_{\mathcal{O}_E} \mathcal{O}_{\breve{E}}$. We will show that $\breve{\mathcal{M}}$ may be covered by étale neighborhoods, each of which is formally locally of finite type and formally étale over $\hat{\mathbb{M}}^{\mathrm{loc}}$ (conventions for terminology regarding formal schemes are reviewed in Appendix A). Here are some example consequences of this comparison between $\breve{\mathcal{M}}$ and $\hat{\mathbb{M}}^{\mathrm{loc}}$.

- If $\mathbb{M}^{\mathrm{loc}}$ is flat over $\operatorname{Spec} \mathcal{O}_E$ then $\breve{\mathcal{M}}$ is flat over $\operatorname{Spf} \mathcal{O}_{\breve{E}}$.
- If $\mathbb{M}^{\mathrm{loc}}$ is smooth over $\operatorname{Spec} \mathcal{O}_E$, then $\breve{\mathcal{M}}$ is formally smooth over $\operatorname{Spf} \mathcal{O}_{\breve{E}}$ (see Proposition A.5).
- Suppose the formal scheme $\breve{\mathcal{M}}$ is p-adic, i.e. p is the ideal of definition of $\breve{\mathcal{M}}$ (e.g. in the Drinfeld example [RZ96, Corollary 3.63]). Let T be any locally Noetherian scheme over $\operatorname{Spf} \mathcal{O}_{\breve{E}}$ (i.e. a locally Noetherian $\mathcal{O}_{\breve{E}}$-scheme on which p is locally nilpotent). Let $\breve{\mathcal{M}}_T = \breve{\mathcal{M}} \times_{\operatorname{Spf} \mathcal{O}_{\breve{E}}} T$ and let $\hat{\mathbb{M}}^{\mathrm{loc}}_T = \hat{\mathbb{M}}^{\mathrm{loc}} \times_{\operatorname{Spf} \mathcal{O}_{\breve{E}}} T$. Then $\breve{\mathcal{M}}_T$ and $\hat{\mathbb{M}}^{\mathrm{loc}}_T$ are locally of finite type T-schemes, and $\breve{\mathcal{M}}_T$ may be covered by étale neighborhoods which are étale over $\hat{\mathbb{M}}^{\mathrm{loc}}_T$. Thus, for every point on $\breve{\mathcal{M}}_T$, there exists a point on $\hat{\mathbb{M}}^{\mathrm{loc}}_T$ with isomorphic local ring for the étale topology (strict Henselization of the usual local ring). In particular, if $\hat{\mathbb{M}}^{\mathrm{loc}}_T$ has property $\mathcal{P}$ then $\breve{\mathcal{M}}_T$ also has property $\mathcal{P}$, where $\mathcal{P}$ can be any of the properties: regular, reduced, normal, Cohen-Macaulay (see [Stacks, Section 025L] or [Stacks, Section 07QL]).

2 Review of Rapoport-Zink spaces and other notation

I first fix some notation and review the setup introduced in the previous talk (Murilo's), with some small modifications. The material in this section may be found in [RZ96, Section

3], in greater generality. For an adic ring O, we write $\mathbf{Sch}/\operatorname{Spf} O$ for the category of locally Noetherian schemes over $\operatorname{Spf} O$. Previously, for a complete discrete valuation ring O of mixed characteristic $(0,p)$, we wrote $\mathbf{Nil}_O$ for the category of schemes on which p is locally nilpotent. Then $\mathbf{Nil}_O$ is the same as $\mathbf{Sch}/\operatorname{Spf} O$. Given $S \in \operatorname{Obj}(\mathbf{Sch}/\operatorname{Spf} O)$ for O a complete discrete valuation ring of mixed characteristic $(0,p)$, we write $\overline{S} := S \times_{\operatorname{Spec} O} \operatorname{Spec} O/(p)$.

Rapoport-Zink data of type EL and PEL are summarized below. Unlabeled items correspond to both the EL and PEL case. Items labeled (EL) or (PEL) apply only for (EL) or (PEL) cases respectively.

Rapoport-Zink data:
Fix data (EL) $(F, B, \mathcal{O}_B, V, \mathcal{L}, \mu, b)$ or (PEL) $(F, B, \mathcal{O}_B, V, \mathcal{L}, \mu, b, *, (\ ,\ ))$ with

- $F/\mathbb{Q}_p$ a finite extension of fields
- B a finite-dimensional central simple algebra over F
- $\mathcal{O}_B$ a maximal $\mathbb{Z}_p$-order in B
    - There necessarily exists a central division algebra D over F, such that $B \cong M_n(D)$ and $\mathcal{O}_B = M_n(\mathcal{O}_D)$, where $\mathcal{O}_D$ is the unique maximal order of D. Fix such identifications, and let $\varpi \in \mathcal{O}_D$ be a uniformizer. Embed D in $M_n(\mathcal{O}_D)$ as diagonal matrices.
    - (PEL) $*$ an anti-involution $b \mapsto b^*$ on B which sends $\mathcal{O}_B$ to itself
- V a finite-dimensional B-module
    - (PEL) $(\ ,\ )$ a non-degenerate alternating $\mathbb{Q}_p$-bilinear pairing, such that $(bv, w) = (vb^*w)$ for all $b \in B$ and $v, w \in V$
- $\mathcal{L} = (\Lambda_i)_{i\in\mathbb{Z}}$ a chain $(\cdots \subsetneq \Lambda_{-1} \subsetneq \Lambda_0 \subsetneq \Lambda_1 \subsetneq \cdots)$ of $\mathcal{O}_B$-stable $\mathbb{Z}_p$-lattices $\Lambda_i \subseteq V$, such that $\Lambda \in \mathcal{L}$ implies $a\Lambda \in \mathcal{L}$ for $a \in B^\times$ normalizing $\mathcal{O}_B$; view $\mathcal{L}$ as a category with objects Λ_i and arrows given by inclusions
    - Under the identification $\mathcal{O}_B \cong M_n(\mathcal{O}_D)$, the normalizer of $\mathcal{O}_B$ is $D^\times \cdot \mathcal{O}_B^\times$.
    - A possible chain is $(\cdots \subsetneq \varpi\Lambda \subsetneq \Lambda \subsetneq \varpi^{-1}\Lambda \subsetneq \cdots)$ for a fixed $\mathcal{O}_B$-stable $\mathbb{Z}_p$-lattice Λ, which is equivalent to the "simple Rapoport-Zink data" of Murilo's talk with one lattice; in this case the functor $\breve{\mathcal{M}}$ below does not depend on the choice of Λ (see [RZ96, pg. 78]).
    - Note: Suppose $a \in B^\times$ normalizes $\mathcal{O}_B$. Given a $\mathcal{O}_B$-module M, write M^a for the $\mathcal{O}_B$-module with underlying abelian group M, and with action of $x \in \mathcal{O}_B$ on M^a given by $a^{-1}xa$ acting on M. Multiplication by a induces a *periodicity isomorphism* $\theta_a \colon \Lambda^a \to a\Lambda$.
    - (PEL) The chain $\mathcal{L}$ is required to be self-dual, in the sense that $\Lambda \in \mathcal{L}$ implies $\Lambda^\vee \in \mathcal{L}$, where $\Lambda^\vee = \{v \in V : (v, w) \in \mathbb{Z}_p \text{ for } w \in \Lambda\}$.
- (EL) $G := \mathrm{GL}_B(V)$, which is an algebraic group over $\mathbb{Q}_p$
- (PEL) $G := \mathrm{GSp}_B(V)$, which is an algebraic group over $\mathbb{Q}_p$
- $\mu \colon \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$ a co-character
- $b \in G(K_0)$ for $K_0 = \breve{\mathbb{Q}}_p$

- E the field of definition of the conjugacy class of μ (also called the "Shimura field" or "reflex field"), with ring of integers $\mathcal{O}_E$; here $E/\mathbb{Q}_p$ is a finite extension of fields
- $\breve{E}$ the completion of the maximal unramified algebraic extension of E, with ring of integers $\mathcal{O}_{\breve{E}}$; the residue field is $k = \overline{\mathbb{F}}_p$

satisfying the conditions

- the isocrystal $(V \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p, b(\mathrm{id} \otimes \sigma))$ has slopes in the interval $[0,1]$; here σ is the Frobenius automorphism of $\breve{\mathbb{Q}}_p$
- (μ, b) is admissible
- there is a weight decomposition $V \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p = V_0 \oplus V_1$ for the action of μ, where V_0 has weight 0 and V_1 has weight 1
- the composition $\mathbb{G}_{m,\overline{\mathbb{Q}}_p} \xrightarrow{\mu} G_{\overline{\mathbb{Q}}_p} \xrightarrow{c} \mathbb{G}_{m,\overline{\mathbb{Q}}_p}$ where c is the *similitude factor* for G acting on V with respect to the pairing $(\ ,\ )$, i.e. $c(g)(v,w) = (gv, gw)$ for $g \in G(R)$, $v, w \in V \otimes_{\mathbb{Q}_p} R$ for $\mathbb{Q}_p$-algebras R.

Rapoport-Zink spaces: Fix Rapoport-Zink data of either PEL or EL type. Suppose there exists a p-divisible group $\mathbb{X}$ over $\operatorname{Spec} \overline{\mathbb{F}}_p$ whose isocrystal is identified with the isocrystal $(V \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p, b(\mathrm{id} \otimes \sigma))$. Fix such an $\mathbb{X}$ and such an identification of isocrystals; this is a "framing object". Last time, we discussed a functor

$$\breve{\mathcal{M}} \colon (\mathbf{Sch}/\operatorname{Spf} \mathcal{O}_{\breve{E}})^{\mathrm{op}} \to \mathbf{Set}$$

such that, for $S \in \mathrm{Obj}(\mathbf{Sch}/\operatorname{Spf} \mathcal{O}_{\breve{E}})$, an element of $\breve{\mathcal{M}}(S)$ consists of tuples $(X_\Lambda, \rho_\Lambda)_{\Lambda \in \mathcal{L}}$ (in both the EL and PEL cases) up to a suitable notion of isomorphism; with

- $\Lambda \mapsto X_\Lambda$ a functorial map from lattices $\Lambda \in \mathcal{L}$ to p-divisible groups X_Λ over S (i.e. $\Lambda \hookrightarrow \Lambda'$ gives a morphism $X_\Lambda \to X_{\Lambda'}$)
- $\rho_\Lambda \colon \mathbb{X}_{\overline{S}} \dashrightarrow X_{\Lambda, \overline{S}}$ a quasi-isogeny.

We required several conditions in the definition of $\breve{\mathcal{M}}$.

- The induced action of $\mathcal{O}_B$ on X_Λ (by Drinfeld rigidity, as in Vijay's talk), a priori by quasi-isogenies, is given by isogenies.
- (Kottwitz condition) We have an identity of polynomial functions

$$\det_{\mathcal{O}_S}(a; \operatorname{Lie} X_\Lambda) = \det_{\overline{\mathbb{Q}}_p}(a; V_0)$$

  for all $a \in \mathcal{O}_B$ and all $\Lambda \in \mathcal{L}$ (see [RZ96, §3.23(a)]).
- There exist polarizations $X_\Lambda \to X_\Lambda^\vee$ suitably compatible with the polarized isocrystal $(V \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p, b(\mathrm{id} \otimes \sigma))$.
- Certain compatibility relations between data $(X_\Lambda, \rho_\Lambda)$ for varying $\Lambda \in \mathcal{L}$ are satisfied.

See [RZ96, Definition 3.21] and [RZ96, §3.23(c),(d)] for further details.

We saw that $\breve{\mathcal{M}}$ is representable by a formal scheme, which is formally locally of finite type over $\operatorname{Spf} \mathcal{O}_{\breve{E}}$. We call $\breve{\mathcal{M}}$ a Rapoport-Zink space, of type EL or PEL.

3 Local models

Fix a Rapoport-Zink space $\breve{\mathcal{M}}$ as in Section 2, of type EL or PEL. In this section, we construct the associated local model $\mathbb{M}^{\text{loc}}$. We give several explicit examples of local models in Section 3.2.

3.1 Definition

We define the local models $\mathbb{M}^{\text{loc}}$ functorially first, and treat the representability afterwards.

Definition 3.1 ([RZ96, Definition 3.27])**.** The *local model* associated to $\breve{\mathcal{M}}$ is the functor

$$\mathbb{M}^{\text{loc}} \colon (\mathbf{Sch}/\operatorname{Spec}\mathcal{O}_E)^{\text{op}} \to \mathbf{Set}$$

such that, for an $\mathcal{O}_E$-scheme S, an element of $\mathbb{M}^{\text{loc}}(S)$ is a set of surjections

$$(\varphi_\Lambda \colon \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S \twoheadrightarrow t_\Lambda)_{\Lambda \in \mathcal{L}}$$

with each t_Λ a finite locally free $\mathcal{O}_S$-module, satisfying the following conditions.

(i) The $\mathcal{O}_B$-action on each $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ descends (necessarily uniquely) to t_Λ along φ_Λ.

(ii) The surjections φ_Λ are functorial in Λ; i.e. if $\Lambda \subseteq \Lambda'$ is an inclusion of lattices in $\mathcal{L}$, the map of $\mathcal{O}_B \otimes_{\mathbb{Z}_p} \mathcal{O}_S$-modules $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S \to \Lambda' \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ descends (necessarily uniquely) to a map $t_\Lambda \to t_{\Lambda'}$ along φ_Λ and $\varphi_{\Lambda'}$. Moreover, if $a \in B^\times$ normalizes $\mathcal{O}_B$, the periodicity isomorphism $\theta_a \otimes id \colon \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S \to a\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ descends to an isomorphism of $\mathcal{O}_B \otimes_{\mathbb{Z}_p} \mathcal{O}_S$-modules $t_\Lambda^a \xrightarrow{\sim} t_{a\Lambda}$ (necessarily uniquely) along φ_Λ and $\varphi_{a\Lambda}$.[1]

(iii) (Kottwitz condition) We have an identity of polynomial functions

$$\det_{\mathcal{O}_S}(a; t_\Lambda) = \det_{\overline{\mathbb{Q}}_p}(a; V_0)$$

for all $a \in \mathcal{O}_B$ and all $\Lambda \in \mathcal{L}$ (see [RZ96, §3.23(a)]).

(iv) (PEL) For each Λ, the composition

$$t_\Lambda^\vee \xrightarrow{\varphi_\Lambda^\vee} (\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S)^\vee \xrightarrow[\sim]{(\ ,\ )} \Lambda^\vee \otimes_{\mathbb{Z}_p} \mathcal{O}_S \xrightarrow{\varphi_{\Lambda^\vee}} t_{\Lambda^\vee}$$

is zero.

Remark 3.2. It may be more precise to view $\mathbb{M}^{\text{loc}}$ as being associated to the Rapoport-Zink data underlying $\breve{\mathcal{M}}$, rather than $\breve{\mathcal{M}}$ itself. Furthermore, the definition of $\mathbb{M}^{\text{loc}}$ does not depend on the element $b \in G(K_0)$ in the Rapoport-Zink data.

Remark 3.3. The flatness conjecture of Rapoport-Zink for local models [RZ96, pg. 95] has turned out to be false. This led Pappas and Rapoport to introduce a modified notion of local model. The local models of Definition 3.1 are the original ones defined by Rapoport-Zink, and are sometimes called "naive local models". See the survey [PRS13].

[1] This periodicity condition appears in the definition of local models given in [SW20, Definition 21.6.8]. They remark that the condition seems to have been overlooked by Rapoport and Zink in [RZ96, Definition 3.27].

Remark 3.4. Condition (iii) of Definition 3.1, i.e. the Kottwitz condition, implies that $\operatorname{rank} t_\Lambda = \dim V_0$ for all Λ. Indeed, for any $a \in \mathbb{Z}_p^\times$, the Kottwitz condition implies

$$a^{\operatorname{rank} t_\Lambda} = a^{\dim V_0}$$

for all Λ. In fact, if $\mathcal{O}_B = \mathbb{Z}_p$, we see that the rank condition $\operatorname{rank} t_\Lambda = \dim V_0$ for all Λ is equivalent to the Kottwitz condition.

I claim that the local model $\mathbb{M}^{\text{loc}}$ is represented by a projective $\mathcal{O}_E$-scheme. Indeed, recall the Grassmanian functor $\operatorname{Gr}(d, n) \colon \mathbf{Sch}^{\text{op}} \to \mathbf{Set}$; an S-point of $\operatorname{Gr}(d, n)$ is a surjection $\mathcal{O}_S^{\oplus n} \twoheadrightarrow \mathcal{E}$ for a locally free $\mathcal{O}_S$-module of rank d (up to isomorphisms in the variable $\mathcal{E}$). In the situation of Definition 3.1, each φ_Λ is an S-point of $\operatorname{Gr}(\dim V_0, \dim V)$.

Suppose $\Lambda_0 \subsetneq \Lambda_1 \subsetneq \cdots \subsetneq \Lambda_r = \varpi^{-1}\Lambda_0$ is a "period" of the chain Λ (as in [RZ96, pg. 70]). The map

$$\begin{aligned} \mathbb{M}^{\text{loc}} &\hookrightarrow \textstyle\prod_{i=0}^r \operatorname{Gr}(\dim V_0, \dim V) \\ (\varphi_\Lambda)_{\Lambda \in \mathcal{L}} &\longmapsto (\varphi_{\Lambda_0}, \ldots, \varphi_{\Lambda_{r-1}}) \end{aligned}$$

exhibits $\mathbb{M}^{\text{loc}}$ as a subfunctor of a product of Grassmanians (upon choosing a basis for each Λ_i). Since each $\operatorname{Gr}(\dim V_0, \dim V)$ is a projective $\mathcal{O}_E$-scheme, it suffices to check that each of the conditions (i) - (iv) in Definition 3.1 is a closed condition in $\prod_{i=0}^r \operatorname{Gr}(\dim V_0, \dim V)$. Each of the conditions is "linear algebraic", e.g. condition (i) is the statement that each $\ker \varphi_\Lambda$ is stable under the action of $\mathcal{O}_B$. I omit the explicit check (as do Rapoport-Zink) that each of (i) - (iv) is a closed condition. However, we will see how conditions (i) - (iv) work in concrete examples in the next section.

3.2 Examples

In this section, we calculate a few concrete examples of local models $\mathbb{M}^{\text{loc}}$. As in Remark 3.2, the local models $\mathbb{M}^{\text{loc}}$ do not depend on the choice of $b \in G(K_0)$ in the formulation of Rapoport-Zink data. We thus do not describe b in this section. Some of the examples below generalize under the same name beyond the situations considered.

Example 3.5 (Lubin-Tate example)**.** See [PRS13, §2.1].

This is an example of EL type. For this Rapoport-Zink data, we take $F = B = \mathbb{Q}_p$, so $\mathcal{O}_B = \mathbb{Z}_p$. Let $n \geq 1$, $V = \mathbb{Q}_p^n$, $\Lambda = \mathbb{Z}_p^n$, and $\mathcal{L} = (p^m\Lambda)_{m \in \mathbb{Z}}$. Take $\mu(t) = (t, \ldots, t, 1)$, so that the 0 weight-space $V_0 \subseteq V$ has dimension 1. The reflex field is $E = \mathbb{Q}_p$. As described in Section 3.1, we view $\mathbb{M}^{\text{loc}}$ as a subfunctor

$$\mathbb{M}^{\text{loc}} \hookrightarrow \operatorname{Gr}(1, n) \cong \mathbb{P}^{n-1}.$$

We need to describe each of the conditions (i) - (iv) of Definition 3.1 on $\mathbb{P}^{n-1}$. Since $\mathcal{O}_B = \mathbb{Z}_p$, condition (i) in Definition 3.1 is automatic. Condition (ii) is then also automatic in this situation, since $\mathcal{L}$ consists of only one lattice up to scalar. Condition (iv) does not apply, since the data is of EL type. Since $\mathcal{O}_B = \mathbb{Z}_p$, condition (iii) (the Kottwitz condition) is equivalent to the rank condition of Remark 3.4. This is again automatic on $\mathbb{P}^{n-1}$. So the above map $\mathbb{M}^{\text{loc}} \to \mathbb{P}^{n-1}$ is an isomorphism.

More generally, if we consider a co-character $\mu(t) = (t, \ldots, t, 1, \ldots, 1)$ such that the 0 weight-space V_0 has dimension $d \geq 1$, we find $\mathbb{M}^{\text{loc}} \xrightarrow{\sim} \operatorname{Gr}(d, n)$ by a nearly identical argument.

In this case, the local model $\mathbb{M}^{\mathrm{loc}}$ is smooth, so Proposition A.5 below will imply that the Rapoport-Zink spaces $\breve{\mathcal{M}}$ associated to this Rapoport-Zink data (for any choice of $b \in G(K_0)$) are formally smooth over Spf $\mathcal{O}_{\breve{E}}$.

Example 3.6 (A PEL example)**.** See [PRS13, §2.4].

This is an example of PEL type. For this Rapoport-Zink data, we set $F = B = \mathbb{Q}_p$, so $\mathcal{O}_B = \mathbb{Z}_p$. Let $n \geq 1$, $V = \mathbb{Q}_p^{2n}$, $\Lambda = \mathbb{Z}_p^{2n}$, and $\mathcal{L} = (p^m\Lambda)_{m\in\mathbb{Z}}$. Take $\mu(t) = (t, \dots, t, 1, \dots, 1)$, so that the 0 weight-space $V_0 \subseteq V$ has dimension n. Let $(\ , \ )$ be the standard symplectic form on V. The anti-involution $*$ on B is necessarily trivial. The reflex field is $E = \mathbb{Q}_p$. As described in Section 3.1, we view $\mathbb{M}^{\mathrm{loc}}$ as a subfunctor

$$\mathbb{M}^{\mathrm{loc}} \hookrightarrow \mathrm{Gr}(n, 2n)$$

We need to describe each of the conditions (i) - (iv) of Definition 3.1 on $\mathrm{Gr}(n, 2n)$. Since $\mathcal{O}_B = \mathbb{Z}_p$, condition (i) in Definition 3.1 is automatic. Condition (ii) is then also automatic in this situation, since $\mathcal{L}$ consists of only one lattice up to scalar. Since $\mathcal{O}_B = \mathbb{Z}_p$, condition (iii) (the Kottwitz condition) is equivalent to the rank condition of Remark 3.4. This is again automatic on $\mathrm{Gr}(n, 2n)$. We observe that condition (iv) is precisely the closed condition defining the *Lagrangian Grassmanian* $\mathrm{LGr}(n, 2n)$ inside $\mathrm{Gr}(n, 2n)$. Hence we have $\mathbb{M}^{\mathrm{loc}} \xrightarrow{\sim} \mathrm{LGr}(n, 2n)$.

In this case, the local model $\mathbb{M}^{\mathrm{loc}}$ is smooth, so Proposition A.5 below will imply that the Rapoport-Zink spaces $\breve{\mathcal{M}}$ associated to this Rapoport-Zink data (for any choice of $b \in G(K_0)$) are formally smooth over Spf $\mathcal{O}_{\breve{E}}$.

Example 3.7 (Iwahori example)**.** See [PRS13, Example 2.4] or [Hai05, §4.4].

This is an example of EL type. For this Rapoport-Zink data, we set $F = B = \mathbb{Q}_p$, so $\mathcal{O}_B = \mathbb{Z}_p$. Let $n \geq 2$, and set $V = \mathbb{Q}_p^n$. Let $\mu(t) = (t, \dots, t, 1)$ so that the 0 weight-space $V_0 \subseteq V$ has dimension 1. The reflex field is $E = \mathbb{Q}_p$. We impose a nontrivial level structure. Consider a chain $\mathcal{L}$ which has a period of length n, given by

$$\Lambda_0 \subsetneq \Lambda_1 \subsetneq \cdots \subsetneq \Lambda_n = p^{-1}\Lambda_0$$

where the inclusions are index p. It may be helpful to think of index p inclusions of lattices arising from degree p-isogenies of p-divisible groups.

We compute $\mathbb{M}^{\mathrm{loc}}$ for $n = 2$. In this case, we view $\mathbb{M}^{\mathrm{loc}}$ as a subfunctor

$$\begin{aligned} \mathbb{M}^{\mathrm{loc}} &\hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \\ (\varphi_\Lambda)_{\Lambda\in\mathcal{L}} &\longmapsto (\varphi_{\Lambda_0}, \varphi_{\Lambda_1}). \end{aligned}$$

Since $\mathcal{O}_B = \mathbb{Z}_p$, condition (i) in Definition 3.1 is automatic (in the sense that no additional equations are imposed on $\mathbb{P}^1 \times \mathbb{P}^1$, for defining $\mathbb{M}^{\mathrm{loc}}$). Similarly, the Kottwitz condition (condition (iii)) is equivalent to the rank condition of Remark 3.4 because $\mathcal{O}_B = \mathbb{Z}_p$, so no additional equations are imposed from condition (iii) either. So $\mathbb{M}^{\mathrm{loc}}$

We need to describe each of the conditions (i) - (iv) of Definition 3.1 on $\mathrm{Gr}(d, 2d)$. Since $\mathcal{O}_B = \mathbb{Z}_p$, condition (i) in Definition 3.1 is automatic. Condition (ii) is then also automatic in this situation, since $\mathcal{L}$ consists of only one lattice up to scalar. Since $\mathcal{O}_B = \mathbb{Z}_p$, condition (iii) (the Kottwitz condition) is equivalent to the rank condition of Remark 3.4. Writing the inclusion $\Lambda_0 \hookrightarrow \Lambda_1$ in Smith normal form, we see that the conditions defining $\mathbb{M}^{\mathrm{loc}}$ inside

$\mathbb{P}^1 \times \mathbb{P}^1$ are the conditions.

$$\varphi_{\Lambda_1} \circ \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} (\ker \varphi_{\Lambda_0}) = 0$$
$$\varphi_{\Lambda_0} \circ \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} (\ker \varphi_{\Lambda_1}) = 0.$$

Equivalently, $\mathbb{M}^{\text{loc}}(S)$ consists of commutative diagrams

$$\begin{array}{ccccc} \mathcal{O}_S^{\oplus 2} & \xrightarrow{\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}} & \mathcal{O}_S^{\oplus 2} & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}} & \mathcal{O}_S^{\oplus 2} \\ \downarrow\!\!\!\downarrow \varphi_{\Lambda_0} & & \downarrow\!\!\!\downarrow \varphi_{\Lambda_1} & & \downarrow\!\!\!\downarrow \varphi_{\Lambda_0} \\ t_{\Lambda_0} & \longrightarrow & t_{\Lambda_1} & \longrightarrow & t_{\Lambda_0} \end{array}$$

(up to isomorphisms, in the variables t_{Λ_0} and t_{Λ_1}), for t_{Λ_0} and t_{Λ_1} locally free $\mathcal{O}_S$-modules of rank 1.

If (x_0, x_1) and (y_0, y_1) are homogeneous coordinates on the first and second copies of $\mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$ respectively, we compute that $\mathbb{M}^{\text{loc}}$ is defined by the equation $px_1y_0 = x_0y_1$ inside $\mathbb{P}^1 \times \mathbb{P}^1$. Thus $\mathbb{M}^{\text{loc}}$ is not smooth over $\mathbb{Z}_p = \mathcal{O}_E$. The generic fiber is isomorphic to $\mathbb{P}^1_{\mathbb{Q}_p}$, but the special fiber consists of two copies of $\mathbb{P}^1_{\mathbb{F}_p}$ meeting at a point. The neighborhood $x_1 \neq 0$ and $y_0 \neq 0$ of the singularity in special fiber is of the form $\operatorname{Spec} \mathbb{F}_p[x, y]/(xy)$.

In this case, the local model $\mathbb{M}^{\text{loc}}$ is not smooth.

Example 3.8 (Drinfeld example)**.** See [RZ96, 3.76].

This is an example of EL type. Let $d \geq 2$. Let $F = \mathbb{Q}_p$. Let $\tilde{F} = \mathbb{Q}_{p^d}$ be the unique degree d unramified extension of $\mathbb{Q}_p$. The corresponding ring of integers is $\mathcal{O}_{\tilde{F}} = \mathbb{Z}_{p^d}$. Let $B = D$ be the division algebra

$$D = \tilde{F}\langle \varpi \rangle / (\varpi - p, \varpi x - {}^{\sigma}x\varpi \text{ for all } x \in \mathbb{Q}_{p^d}) \quad \text{with}$$
$$\mathcal{O}_D = \mathcal{O}_{\tilde{F}}\langle \varpi \rangle / (\varpi - p, \varpi x - {}^{\sigma}x\varpi \text{ for all } x \in \mathbb{Q}_{p^d})$$

where $x \mapsto {}^{\sigma}x$ is Frobenius on $\tilde{F}$. Set $V = D$, with action by D via left multiplication. Set $\Lambda = \mathcal{O}_D$ and $\mathcal{L} = (\varpi^m \Lambda)_{m \in \mathbb{Z}}$.

We know D is split by $\mathbb{Q}_{p^d}$, i.e. $\mathbb{Q}_{p^d} \otimes_{\mathbb{Q}_p} B \cong M_d(\mathbb{Q}_{p^d})$. In this case, $G \times_{\mathbb{Q}_p} \mathbb{Q}_{p^d} = \mathrm{GL}^{\text{op}}_{n, \mathbb{Q}_{p^d}}$, acting on $V_{\mathbb{Q}_{p^d}}$ by right multiplication. Let $\mu \colon \mathbb{G}_{m, \mathbb{Q}_{p^d}} \to \mathrm{GL}^{\text{op}}_{n, \mathbb{Q}_{p^d}}$ be the co-character $\mu(t) = (t, \ldots, t, 1)$. Here $V_0 \subseteq V_{\mathbb{Q}_{p^d}}$ has dimension d.

We compute $\mathbb{M}^{\text{loc}} \times_{\mathcal{O}_E} \mathbb{Z}_{p^d}$.[2] For $S \in \operatorname{Obj}(\mathbf{Sch}/\mathbb{Z}_{p^d})$, we know $\mathbb{M}^{\text{loc}}(S)$ consists of a surjection $\varphi_\Lambda \colon \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S \twoheadrightarrow t_\Lambda$, where t_Λ is locally free of rank $d = \dim V_0$ (by condition (iii) and Remark 3.4), satisfying the conditions imposed by Definition 3.1.

If condition (i) holds for φ_Λ, the condition (ii) is then automatic, since $\mathcal{L}$ consists of only one lattice Λ up to scalar. Moreover, if condition (i) holds (so that the statement of condition (iii) makes sense), we saw in the previous lecture (Remark 4.3 in Murilo's notes,

[2]Note $\mathbb{Z}_{p^d}$ is contained in $\mathcal{O}_{\breve{E}}$, so we are still able to compute $\mathbb{M}^{\text{loc}} \times_{\mathcal{O}_E} \mathcal{O}_{\breve{E}}$. In Section 4 below, the comparison to Rapoport-Zink spaces $\breve{\mathcal{M}}$ is through $\mathbb{M}^{\text{loc}} \times_{\mathcal{O}_E} \mathcal{O}_{\breve{E}}$.

or [RZ96, 3.23(b)]), that the Kottwitz condition (condition (iii)) in this situation is then equivalent to the requirement that

$$t_\Lambda = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} t_\Lambda^i$$

where each t_{Λ^i} is a locally free $\mathcal{O}_S$-module of rank 1, on which $\mathcal{O}_{\tilde{F}}$ acts as

$$\begin{aligned} \mathcal{O}_{\tilde{F}} &\longrightarrow \mathbb{Z}_{p^d} \\ x &\longmapsto \sigma^{-i} x \sigma^i \end{aligned}$$

where the right-hand side acts as scalars on each t_Λ^i.

Putting this together, we see that an S-point of $\mathbb{M}^{\mathrm{loc}}$ is a set of $\mathcal{O}_{\tilde{F}} \otimes_{\mathbb{Z}_p} \mathcal{O}_S$-linear surjections

$$(\varphi_i \colon \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S \twoheadrightarrow t^i)_{i \in \mathbb{Z}/d\mathbb{Z}}$$

where

(a) Each t^i is a locally free $\mathcal{O}_S$-module of rank 1

(b) $\mathcal{O}_{\tilde{F}}$ acts on t^i by the scalar $\mathcal{O}_{\tilde{F}} \to \mathbb{Z}_{p^d}$ given by $x \mapsto \sigma^{-i} x \sigma^i$ as above

(c) For each solid diagram

$$\begin{array}{ccc} \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S & \xrightarrow{\times \varpi} & \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S \\ \downarrow{\scriptstyle \varphi_i} & & \downarrow{\scriptstyle \varphi_{i+1}} \\ t^i & \dashrightarrow & t^{i+1} \end{array}$$

there is a dotted arrow (necessarily unique) which makes the diagram commute.

The original data φ_Λ and t_Λ may be recovered by setting $\varphi_\Lambda = \bigoplus \varphi_i$ and $t_\Lambda = \bigoplus t^i$.

Moreover, we know $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ is a free $\mathcal{O}_{\tilde{F}} \otimes_{\mathbb{Z}_p} \mathcal{O}_S$-module with basis $1, \varpi, \ldots, \varpi^{d-1}$. Let $\Gamma \subseteq \Lambda$ be the free $\mathbb{Z}_p$-module with basis $1, \varpi, \ldots, \varpi^{d-1}$. We see that specifying $(\varphi_i)_{i \in \mathbb{Z}/d\mathbb{Z}}$ as above is the same as specifying a set of surjections of locally free $\mathcal{O}_S$-modules

$$(\varphi_i' \colon \Gamma \otimes_{\mathbb{Z}_p} \mathcal{O}_S \to t^i)_{i \in \mathbb{Z}/d\mathbb{Z}}$$

where t^i is locally free of rank 1, such that

(c') For each solid diagram

$$\begin{array}{ccc} \Gamma \otimes_{\mathbb{Z}_p} \mathcal{O}_S & \xrightarrow{\times \varpi} & \Gamma \otimes_{\mathbb{Z}_p} \mathcal{O}_S \\ \downarrow{\scriptstyle \varphi_i'} & & \downarrow{\scriptstyle \varphi_{i+1}'} \\ t^i & \dashrightarrow & t^{i+1} \end{array}$$

there is a dotted arrow (necessarily unique) which makes the diagram commute.

So we may view $\mathbb{M}^{\mathrm{loc}} \times_{\mathcal{O}_E} \mathbb{Z}_{p^d}$ as the subfunctor

$$\begin{aligned} \mathbb{M}^{\mathrm{loc}} \otimes_{\mathcal{O}_E} \mathbb{Z}_{p^d} &\hookrightarrow \textstyle\prod_{i \in \mathbb{Z}/d\mathbb{Z}} \mathbb{P}^{d-1} \\ \varphi_\Lambda &\longmapsto (\varphi_i')_{i \in \mathbb{Z}/d\mathbb{Z}} \end{aligned}$$

subject to the condition imposed by (c') above, i.e.

$$\varphi'_{i+1}(\varpi \cdot \ker \varphi'_o) = 0 \quad \text{for } i \in \mathbb{Z}/d\mathbb{Z}.$$

The basis $1, \varpi, \ldots, \varpi^{d-1}$ of Γ gives homogeneous coordinates $(x_0^{(i)}, \ldots, x_{d-1}^{(i)})$ on the i-th copy of $\mathbb{P}^{d-1}$ in the product $\prod_{i \in \mathbb{Z}/d\mathbb{Z}} \mathbb{P}^{d-1}$. Condition (c') may be expressed as a closed condition in terms of these coordinates. For example, when $d = 2$, we find that $\mathbb{M}^{\mathrm{loc}} \times_{\mathcal{O}_E} \mathbb{Z}_{p^d}$ is the closed subscheme of $\mathbb{P}^1 \times \mathbb{P}^1$ defined by the equation

$$p x_0^{(0)} x_0^{(1)} = x_1^{(0)} x_1^{(1)}.$$

In this case, $\mathbb{M}^{\mathrm{loc}}$ is not smooth. Moreover, we see that $\mathbb{M}^{\mathrm{loc}} \times_{\mathcal{O}_E} \mathbb{Z}_{p^d}$ is isomorphic (over $\mathbb{Z}_{p^d}$) to the local model for the $n = 2$ Iwahori case from Example 3.7.

3.3 Unramified Rapoport-Zink data

Definition 3.9 (Unramified Rapoport-Zink data)**.** See [RZ96, 3.82].

Consider EL type Rapoport-Zink data $(F, B, \mathcal{O}_B, V, \mathcal{L}, \mu, b)$ as in Section 2. We say the data is *unramified* if F is an unramified extension of $\mathbb{Q}_p$, we have $B = M_n(F)$, and $\mathcal{L} = (\varpi^m \Lambda)_{m \in \mathbb{Z}}$ for some fixed lattice Λ. For PEL Rapoport-Zink data $(F, B, \mathcal{O}_B, V, \mathcal{L}, \mu, b, *, (\ ,\ ))$ as in Section 2 we say the data is *unramified* if it satisfies the conditions for unramified EL data in addition to the requirement $\Lambda = \Lambda^\vee$.

Remark 3.10. In the more general setup of [RZ96, §3], the definition of unramified expands to include the situation where F is a product of unramified field extensions F_i of $\mathbb{Q}_p$, B is a product of matrix algebras $M_{n_i}(F_i)$, and $\mathcal{L}$ is a product of lattice chains as appearing in the preceding definition.

For the following theorem, see [RZ96, 3.82].

Theorem 3.11 (Kottwitz)**.** *For unramified Rapoport-Zink data, the Rapoport-Zink space $\breve{\mathcal{M}}$ is formally smooth over* $\operatorname{Spf} \mathcal{O}_{\breve{E}}$.

Example 3.12. The Lubin-Tate example (Example 3.5) and the PEL example (Example 3.6) of the preceding section arose from unramified Rapoport-Zink data. We saw explicitly in these cases that $\breve{\mathcal{M}}$ must be formally smooth over $\operatorname{Spf} \mathcal{O}_{\breve{E}}$. On the other hand, the Iwahori example (Example 3.7) and the Drinfeld example (Example 3.8) arose from ramified Rapoport-Zink data. For $n = 2$ (Iwahori) and $d = 2$ (Drinfeld) respectively, we saw that the associated local models $\mathbb{M}^{\mathrm{loc}}$ are not smooth.

4 Relating local models and Rapoport-Zink spaces

In this section, we primarily follow the exposition of [RZ96, Section 3.26-3.35]. Terminology regarding formal schemes is reviewed in Appendix A.

We write $\breve{\mathbb{M}}^{\mathrm{loc}} := \mathbb{M}^{\mathrm{loc}} \times_{\operatorname{Spec} \mathcal{O}_E} \operatorname{Spec} \mathcal{O}_{\breve{E}}$. We also write $\hat{\mathbb{M}}^{\mathrm{loc}}$ for the p-adic completion of $\breve{\mathbb{M}}^{\mathrm{loc}}$. Since $\mathbb{M}^{\mathrm{loc}}$ is a projective $\mathcal{O}_E$-scheme, we know that $\hat{\mathbb{M}}^{\mathrm{loc}}$ is formally locally of finite type over $\operatorname{Spf} \mathcal{O}_{\breve{E}}$.

In this section, we will compare $\breve{\mathcal{M}}$ and $\hat{\mathbb{M}}^{\mathrm{loc}}$ as mentioned in the introduction, Section 1. This will be accomplished via the *local model diagram*

$$\begin{array}{ccc} & \mathcal{N} & \\ {}^{s}\!\nearrow \swarrow_{\pi} & & \searrow^{\tilde{\varphi}} \\ \breve{\mathcal{M}} & & \hat{\mathbb{M}}^{\mathrm{loc}} \end{array}$$

which will have the following properties

- Each of $\mathcal{N}$, $\breve{\mathcal{M}}$, and $\hat{\mathbb{M}}^{\mathrm{loc}}$ are locally Noetherian formal schemes, formally of finite type over $\operatorname{Spf} \mathcal{O}_{\breve{E}}$, and each of $s, \pi, \tilde{\varphi}$ are formally locally of finite type
- The map π is smooth, and the map $\tilde{\varphi}$ is formally smooth.
- The map s is a section of π, defined on a suitable étale neighborhood of of $\breve{\mathcal{M}}$
- For a suitable smooth affine group scheme $\mathcal{G}$ over $\mathbb{Z}_p$, π is a $\mathcal{G}_{\operatorname{Spf} \mathcal{O}_{\breve{E}}}$-torsor and $\tilde{\varphi}$ is $\mathcal{G}_{\operatorname{Spf} \mathcal{O}_{\breve{E}}}$-equivariant.
- We may cover $\breve{\mathcal{M}}$ by étale neighborhoods admitting sections s, so that each map $\tilde{\varphi} \circ s$ is formally étale.

These properties imply the consequences claimed in Section 1. The rest of this section describes how to set up the local model diagram, and how to establish the described properties above. Suppose $S \in \operatorname{Obj}(\mathbf{Sch}/\operatorname{Spf} \mathcal{O}_{\breve{E}})$. In the Danielle's talk, we discussed the crystal $\mathbb{D}(X)$ associated to a p-divisible group X over S. This is sheaf on the crystalline site over S, and we write $\mathbb{D}(X)_{S'}$ for its value on an object $(S \to S')$ of the crystalline site over S (with $S \to S'$ understood). See also [Mes72] or [Wan09, §1].

Definition 4.1. Define a functor $\mathcal{N} \colon (\mathbf{Sch}/\operatorname{Spf} \mathcal{O}_{\breve{E}})^{\mathrm{op}} \to \mathbf{Set}$ such that an element of $\mathcal{N}(S)$ is set of triples $(X_\Lambda, \rho_\Lambda, \gamma_\Lambda)_{\Lambda \in \mathcal{L}}$, where $(X_\Lambda, \rho_\Lambda)_{\Lambda \in \mathcal{L}} \in \breve{\mathcal{M}}(S)$ and γ_Λ is an isomorphism of $\mathcal{O}_B \otimes_{\mathbb{Z}_p} \mathcal{O}_S$-modules

$$\gamma_\Lambda \colon M_\Lambda \xrightarrow{\sim} \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S$$

where $M_\Lambda := \mathbb{D}(X_\Lambda)_S$. There are further compatibility conditions on the trivializations γ_Λ which I omit (e.g. functoriality in Λ, compatibility with periodicity isomorphisms, and compatibility with polarizations in the PEL case); see [RZ96, Definition 3.28].

We will see shortly that $\mathcal{N}$ is represented by a locally Noetherian formal scheme. There is a forgetful map

$$\begin{aligned} \mathcal{N} &\xrightarrow{\quad \pi \quad} \breve{\mathcal{M}} \\ (X_\Lambda, \rho_\Lambda, \gamma_\Lambda)_{\Lambda \in \mathcal{L}} &\longmapsto (X_\Lambda, \rho_\Lambda)_{\Lambda \in \mathcal{L}}. \end{aligned}$$

Lifting an element $(X_\Lambda, \rho_\Lambda) \in \breve{\mathcal{M}}(S)$ to an element of $\mathcal{N}(S)$ is the same as giving a trivialization $\gamma_\Lambda \colon M_\Lambda \xrightarrow{\sim} \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ (of (polarized) multi-chains). By [RZ96, Theorem 3.11] and [RZ96, Theorem 3.16], such trivializations exist étale locally on $\breve{\mathcal{M}}$ (in fact, Zariski locally in the EL case). This implies that π admits sections s étale locally.

Next, consider the functor $\mathbf{Sch}/\mathbb{Z}_p \to \mathbf{Set}$ given by

$$S \mapsto \operatorname{Aut}(\{\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S\})$$

where the automorphisms are as chains (polarized, in the PEL case) in the sense of [RZ96, Definition 3.6] or [RZ96, Corollary 3.7]. This functor is representable by a smooth affine group scheme $\mathcal{G}$ over $\mathbb{Z}_p$, by [RZ96, Theorem 3.11] and [RZ96, Theorem 3.16].

Example 4.2. For EL Rapoport-Zink data with $\mathcal{L} = (\varpi^m \mathcal{L})_{m \in \mathbb{Z}}$, we have $\mathcal{G} = \mathrm{GL}_{\mathcal{O}_B}(\Lambda)$.

Observe that $\mathcal{G}$ acts simply transitively on the fibers of $\mathcal{N} \xrightarrow{\pi} \breve{\mathcal{M}}$. Since π admits sections étale locally, we conclude that $\mathcal{N}$ is a $\mathcal{G}$-torsor. Using affine-ness of $\mathcal{G}$, we see that $\mathcal{N}$ is representable by a locally Noetherian formal scheme. Indeed, over a scheme, fpqc torsors for affine group schemes are representable by schemes, essentially by descent for affine morphisms [Stacks, Section 0244]. See also [Mil80, Theorem III.4.3]. A small (omitted) argument shows that, over a locally Noetherian formal scheme, an étale torsor for a finite type affine group scheme must be representable by a locally Noetherian formal scheme. Since $\mathcal{G}$ is smooth, we also find that π is a smooth morphism of locally Noetherian formal schemes (in particular, formally locally of finite type).

Next, we define

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\qquad\qquad\tilde{\varphi}\qquad\qquad} & \breve{\mathbb{M}}^{\mathrm{loc}} \\ (X_\Lambda, \rho_\Lambda, \gamma_\Lambda) & \longmapsto & (\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S \xrightarrow[\sim]{\gamma_\Lambda^{-1}} M_\Lambda \twoheadrightarrow \mathrm{Lie}\, X_\Lambda) \end{array}$$

where $\mathrm{Lie}\, X_\Lambda$ is t_Λ in the local model definition (Definition 3.1), and where the surjection $M_\Lambda \twoheadrightarrow \mathrm{Lie}\, X_\Lambda$ arises from the canonical exact sequence

$$0 \to (\mathrm{Lie}\, X_\Lambda^\vee)^\vee \to M_\Lambda \to \mathrm{Lie}\, X_\Lambda \to 0$$

(see [Wan09, §1]). Since $\mathcal{N}$ is a formal scheme over $\mathrm{Spf}\, \mathcal{O}_{\breve{E}}$, we see that $\tilde{\varphi}$ factors through the p-adic completion $\hat{\mathbb{M}}^{\mathrm{loc}}$ of $\breve{\mathbb{M}}^{\mathrm{loc}}$. Since $\mathcal{N}$ and $\hat{\mathbb{M}}^{\mathrm{loc}}$ are both formally locally of finite type over the locally Noetherian formal scheme $\mathrm{Spf}\, \mathcal{O}_{\breve{E}}$, we know that $\tilde{\varphi}$ is also formally locally of finite type. Moreover, the group $\mathcal{G}$ acts on $\breve{M}^{\mathrm{loc}}$ as $(\varphi_\Lambda)_{\Lambda \in \mathcal{L}} \mapsto (\varphi_\Lambda \circ g^{-1})_{\Lambda \in \mathcal{L}}$ for $g \in \mathcal{G}$.

The key-input for further study of $\tilde{\varphi}$ is Grothendieck-Messing theory, e.g. as in [Wan09, §1].

Theorem 4.3 (Grothendieck-Messing)**.** *Let S be a scheme on which p is locally nilpotent. Let $S \hookrightarrow S'$ be nilpotent thickening, with locally nilpotent divided powers. Let X be a p-divisible group over S. Giving a p-divisible group X' over S' which lifts X is equivalent to giving a surjection of finite locally free $\mathcal{O}'_S$-modules*

$$\varphi' \colon \mathbb{D}(X)_{S'} \to t'$$

which recovers the canonical surjection $\mathbb{D}(X)_S \to \mathrm{Lie}\, X$ upon restricting to S, using the canonical isomorphism

$$\mathbb{D}(X)_{S'} \otimes_{\mathcal{O}_{S'}} \mathcal{O}_S \cong \mathbb{D}(X)_S$$

from the crystalline site. Moreover, we may identify φ' with the canonical surjection $\mathbb{D}(X')_{S'} \to \mathrm{Lie}\, X'$.

Proof. See e.g. [Mes72, §V]. □

Lemma 4.4. *The map $\tilde{\varphi} \colon \mathcal{N} \to \hat{\mathbb{M}}^{\mathrm{loc}}$ is formally smooth.*

Proof. See [RZ96, pg. 90]. The key input is Grothendieck-Messing theory. We wish to show that, for every solid commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & \mathcal{N} \\ \big\downarrow & \overset{\alpha'}{\nearrow} & \big\downarrow \tilde{\varphi} \\ S' & \xrightarrow{\beta} & \hat{\mathbb{M}}^{\mathrm{loc}} \end{array}$$

with $S \to S'$ a first order thickening, there exists a dotted arrow to that the preceding diagram commutes. Write $\alpha = (X_\Lambda, \rho_\Lambda, \gamma_\Lambda)_{\Lambda \in \mathcal{L}} \in \mathcal{N}(S)$ and $\beta = (\varphi'_\Lambda \colon \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{S'} \to t'_\Lambda)_{\Lambda \in \mathcal{L}} \in \hat{\mathbb{M}}^{\mathrm{loc}}(S')$. By Grothendieck-Messing theory, lifting α to $\alpha' = (X'_\Lambda, \rho'_\Lambda, \gamma'_\Lambda)_{\Lambda \in \mathcal{L}} \in \mathcal{N}(S')$ over β is the same as finding a system of arrows γ'_Λ which are isomorphisms of (polarized, if PEL) chains and which make the following diagrams commute for all Λ:

$$\begin{array}{ccc} \mathbb{D}(X_\Lambda)_{S'} & \overset{\gamma'_\Lambda}{\underset{\sim}{\dashrightarrow}} & \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{S'} \\ \big\downarrow & & \big\downarrow \\ \mathbb{D}(X_\Lambda)_S & \overset{\gamma_\Lambda}{\underset{\sim}{\longrightarrow}} & \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S \end{array} .$$

Note that the quasi-isogenies ρ_Λ lifts uniquely to a quasi-isognies ρ'_Λ by Drinfeld rigidity, as in Vijay's talk or [RZ96, pg. 52]. When $\mathcal{L}$ consists of a single lattice Λ up to scalar, we may lift γ_Λ^{-1} to $\gamma_\Lambda'^{-1}$ (non-uniquely) by free-ness of $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{S'}$ (any such lift is surjective by Nakayama's lemma, and hence an isomorphism because $\mathbb{D}(X_\Lambda)_{S'}$ is locally free). In the general case, one should also ensure compatibility of the maps γ'_Λ with functoriality in Λ and compatibility with periodicity isomorphisms, for the chain $\mathcal{L}$. I omit this. ☐

It remains only to see that there exist étale local sections s of $\mathcal{N} \xrightarrow{\pi} \breve{\mathcal{M}}$ with $\tilde{\varphi} \circ s$ formally étale. The content of [RZ96, Proposition 3.33] is precisely that such sections of π exist étale locally near any point of $\breve{\mathcal{M}}$. Below, I sketch the broad ideas of the proof.

It is enough to show that every closed point $x \in \breve{\mathcal{M}}$ admits an etale neighborhood $\mathcal{U}$ and a section $s \colon \mathcal{U} \to \mathcal{N}_\mathcal{U}$ (where $\mathcal{N}_\mathcal{U} = \mathcal{N} \times_{\breve{\mathcal{M}}} \mathcal{U}$) such $\tilde{\varphi} \circ s$ formally étale (with abuse of notataion, we consider $\tilde{\varphi}$ as a map from $\mathcal{N}_\mathcal{U}$ as well). Suppose s is some section as above; we will later give a condition on s that ensures $\tilde{\varphi} \circ s$ is formally étale.

The first part of the proof of [RZ96, Proposition 3.33] reduces étaleness of $\tilde{\varphi} \circ s$ in a Zariski neighborhood of x to the requirement that every solid commutative diagram

$$\begin{array}{ccccc} \operatorname{Spec} k & \xrightarrow{x} & \mathcal{U} & & \\ & & & \overset{s}{\searrow} & \\ \big\downarrow & \nearrow & \big\downarrow \tilde{\varphi}\circ s & & \mathcal{N}_\mathcal{U} \\ & & & \underset{\tilde{\varphi}}{\swarrow} & \\ \operatorname{Spec} k[x]/(x^2) & \longrightarrow & \hat{\mathbb{M}}^{\mathrm{loc}} & & \end{array} \qquad (4.1)$$

admits a unique dotted arrow which makes the diagram commute. Roughly speaking, this is the condition that $\tilde{\varphi} \circ s$ is a "bijection on tangent vectors" (and certainly this condition must be satisfied if $\tilde{\varphi} \circ s$ is to be formally étale).

Next, I introduce some non-standard terminology. Consider $\alpha = (X_\Lambda, \rho_\Lambda, \gamma_\Lambda)_{\Lambda \in \mathcal{L}} \in \mathcal{N}(k)$. Let $\alpha' = (X'_\Lambda, \rho'_\Lambda, \gamma'_\Lambda)_{\Lambda \in \mathcal{L}} \in \mathcal{N}(k[x]/(x^2))$ be a tangent vector at α (i.e. α' recovers α under the map $k[x]/(x^2) \to k$). We say that α' is *distinguished* if the diagram

$$\begin{array}{ccc} \mathbb{D}(X'_\Lambda)_{k[x]/(x^2)} & \xrightarrow{\gamma'_\Lambda} & \Lambda \otimes_{\mathbb{Z}_p} k[x]/(x^2) \\ \uparrow & & \uparrow \\ \mathbb{D}(X_\Lambda)_k & \xrightarrow{\gamma_\Lambda} & \Lambda \otimes_{\mathbb{Z}_p} k \end{array}$$

commutes. The right-hand vertical arrow is induced by the inclusion $k \to k[x]/(x^2)$, and the left-hand vertical arrow is induced by the canonical isomorphism $D(X_\Lambda)_k \otimes_k k[x]/(x^2) \xrightarrow{\sim} \mathbb{D}(X'_\Lambda)_{k[x]/(x^2)}$ arising from the crystalline nature of $\mathbb{D}(X_\Lambda)$.

Suppose α has image β and ξ in $\hat{\mathbb{M}}^{\mathrm{loc}}$ and $\breve{\mathcal{M}}$ respectively. We find that any tangent vector at β lifts uniquely to a distinguished tangent vector at α (by Grothendieck-Messing theory), and also that any tangent vector at ξ lifts uniquely to a distinguished tangent vector at α. The requirement of Diagram (4.1) is thus satisfied if $s\colon \mathcal{U} \to \mathcal{N}_{\mathcal{U}}$ takes any tangent vector at x to a distinguished tangent vector. The existence of such an étale local section s near x is given by [RZ96, 3.32]

A Formal schemes and formal smoothness

A.1 Review of terminology

Suppose $f\colon \mathcal{X} \to \mathcal{Y}$ is a morphism of locally Noetherian formal schemes.

We say $f\colon \mathcal{X} \to \mathcal{Y}$ is *formally locally of finite type* ([RZ96, Definition 2.3]) if $\mathcal{X}_{\mathrm{red}} \to \mathcal{Y}_{\mathrm{red}}$ is a locally finite type morphism of schemes. This property is local on the base, and if $\mathcal{Y} = \operatorname{Spf} A$, is equivalent to the statement that $\mathcal{X}$ is locally of the form $\operatorname{Spf} A\langle T_1, \ldots, T_n\rangle[\![x_1, \ldots, x_m]\!]/I$ for some ideal I (where $A\langle T_1, \ldots, T_n\rangle$ denotes the restricted power series ring). See also [AJP07, §1].

In [RZ96, 2.2], Rapoport-Zink require that étale morphisms and smooth morphisms of formal schemes are representable by schemes. Here $f\colon \mathcal{X} \to \mathcal{Y}$ is *representable by schemes* if for every map $T \to \mathcal{Y}$ with T a scheme, the space $\mathcal{X} \times_{\mathcal{Y}} T$ is a scheme. This is equivalent to the requirement that f is *adic*, i.e. some (equivalently, any) ideal of definition for $\mathcal{Y}$ generates an ideal of idefinition for $\mathcal{X}$, i.e. $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_{\mathrm{red}}$ is a scheme. We say that f is smooth (resp. étale) if for any morphism $T \to \mathcal{Y}$ with T a scheme, the map $\mathcal{X} \times_{\mathcal{Y}} T \to T$ is a smooth (resp. étale) morphism of schemes. If f is smooth or étale, then f is necessarily formally locally of finite type.

Note that the usage of étale and smooth described in the previous paragraph is different from the usage in [AJP07; AJP09], which I will cite below.[3] For example, if k is a field (with the discrete topology), the map $\operatorname{Spf} k[\![x]\!] \to \operatorname{Spec} k$ is not smooth in the sense of the preceding paragraph (but the map is formally smooth, as defined below).

Giving an étale map $\mathcal{U} \to \mathcal{Y}$ is equivalent to giving an étale map of schemes $U \to \mathcal{Y}_{\mathrm{red}}$, essentially by topological invariance of the small étale site [Stacks, Section 04DY] or [EGAIV4, Théorème 18.1.2].

Formal smoothness and formal étaleness is defined as for schemes: we say that $f\colon \mathcal{X} \to \mathcal{Y}$

[3]For [AJP07; AJP09], a morphism f of locally Noetherian formal schemes is said to be smooth (resp. étale) if and only if f is formally locally of finite type and formally smooth (resp. étale).

is *formally smooth* (resp. *formally étale*) if for every solid commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \longleftarrow & T \\ {\scriptstyle f}\big\downarrow & \nwarrow & \big\downarrow \\ \mathcal{Y} & \longleftarrow & T' \end{array}$$

with $T \to T'$ a first order thickening of schemes, there exists at least one (resp. exists exactly one) dotted arrow making the diagram commute. Smooth morphisms for formal schemes as described above are formally smooth, and similarly étale morphisms for formal schemes as described above are formally étale (follows from the analogous statements for schemes).

We say $f\colon \mathcal{X} \to \mathcal{Y}$ is *flat* if for every point $x \in \mathcal{X}$ with $y = f(x)$, the induced map of local rings $\mathcal{O}_{\mathcal{Y},y} \to \mathcal{O}_{\mathcal{X},x}$ is flat. If f is formally locally of finite type and formally smooth then f is flat [AJP07, Proposition 4.8]. The morphism f is *faithfully flat* if it is flat and surjective as a map of topological spaces.

A.2 Formal smoothness

The purpose of this section is to give a proof of Proposition A.5 below, which allows one to relate smoothness of $\mathbb{M}^{\mathrm{loc}}$ to formal smoothness of $\breve{\mathcal{M}}$. The analogous result for schemes is [Stacks, Lemma 02K5], but I did not find a reference for the version for formal schemes.

The proof below proceeds by using differentials for formal schemes, and mimicking the proof for schemes. Consider a formally locally finite type morphism of locally Noetherian formal schemes $f\colon \mathcal{X} \to \mathcal{Y}$. Suppose first that $\mathcal{X} = \operatorname{Spf} B$ and $\mathcal{Y} = \operatorname{Spf} A$. We may form the usual module of differentials $\Omega_{B/A}$, forgetting the topologies on A and B. Then we let $\Omega_{\operatorname{Spf} B/\operatorname{Spf} A}$ the coherent sheaf on $\operatorname{Spf} B$ corresponding to $\hat{\Omega}_{B/A}$, which denotes the completion of $\Omega_{B/A}$ with respect to any ideal of definition for B. This construction globalizes: for general locally Noetherian formal schemes $\mathcal{X}, \mathcal{Y}$, we obtain the sheaf of differentials $\Omega_{\mathcal{X}/\mathcal{Y}}$. This is a coherent $\mathcal{O}_{\mathcal{X}}$-module, and behaves similarly to the usual sheaf of differentials for schemes. See [AJP07] and also [EGAIV1, §0.20]. I general, I found [AJP05; AJP07; AJP09] to be helpful references.

Lemma A.1. *Let k be an algebraically closed field and let $f\colon \mathcal{X} \to \operatorname{Spec} k$ be a formally locally finite type morphism of formal schemes. Let $x \in \mathcal{X}$ be a closed point. Then we have $\dim \Omega_{\mathcal{X}/k} \otimes k(x) = \dim \mathcal{O}_{\mathcal{X},x}$ if and only if $\mathcal{O}_{\mathcal{X},x}$ is regular.*

Proof. We have $\Omega_{\mathcal{X}/k} \otimes k(x) = \Omega_{\operatorname{Spf} \mathcal{O}_{\mathcal{X},x}/k} \otimes k(x) = \Omega_{\mathcal{O}_{\mathcal{X},x}/k} \otimes k(x)$. Since $\mathcal{O}_{\mathcal{X},x}$ is a Noetherian local ring which contains a copy of its residue field k, we have

$$\dim_k \Omega_{\mathcal{O}_{\mathcal{X},x}/k} \otimes k(x) = \dim_k \mathfrak{m}_x/\mathfrak{m}_x^2$$

where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{\mathcal{X},x}$. □

Lemma A.2. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a formally locally finite type morphism of locally Noetherian formal schemes. If f is formally smooth, then $\Omega_{\mathcal{X}/\mathcal{Y}}$ is finite locally free on $\mathcal{X}$.*

Proof. See [AJP07, Proposition 4.8]. □

Lemma A.3. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a formally locally finite type morphism of locally Noetherian formal schemes. Suppose $\mathcal{Z} \to \mathcal{Y}$ is a formally locally finite type faithfully flat morphism of locally Noetherian formal schemes. Then f is formally smooth if and only if $f_{\mathcal{Z}}\colon \mathcal{X}_{\mathcal{Z}} \to \mathcal{Y}_{\mathcal{Z}}$ is formally smooth.*

Proof. Being formally smooth is preserved under base-change, so if f is formally smooth then $f_{\mathcal{Z}}$ is also formally smooth.

Conversely, suppose that $f_{\mathcal{Z}}$ is formally smooth. Since $\mathcal{X}_{\mathcal{Z}} \to \mathcal{X}$ is faithfully flat, Lemma A.2 implies that $\Omega_{\mathcal{X}/\mathcal{Y}}$ is finite locally free. The lemma claim is local on $\mathcal{X}$ and $\mathcal{Y}$, so assume $\mathcal{Y} = \operatorname{Spf} A$ and $\mathcal{X} = \operatorname{Spf} A\langle T_1, \ldots, T_n\rangle[\![x_1, \ldots, x_m]\!]/I$. Write $\mathcal{W} = \operatorname{Spf} A\langle T_1, \ldots, T_n\rangle[\![x_1, \ldots, x_m]\!]/I$, which is formally smooth over $\operatorname{Spf} A$. Let $i\colon \mathcal{X} \to \mathcal{W}$ be the given closed immersion, and let $\mathscr{I}$ be the ideal sheaf of $\mathcal{X}$ on $\mathcal{W}$. There is an exact sequence

$$\mathscr{I}/\mathscr{I}^2 \to i^*\Omega_{\mathcal{W}/\mathcal{Y}} \to \Omega_{\mathcal{X}/\mathcal{Y}} \to 0.$$

In our situation, this is left exact if and only if $\mathcal{X} \to \mathcal{Y}$ is formally smooth (see [AJP07, Corollary 4.15]). This exactness may be checked after faithfully flat base-change, and formation of the preceding exact sequence also commutes with flat base-change. □

Lemma A.4. *Let k be an algebraically closed field, and let $f\colon \mathcal{X} \to \operatorname{Spec} k$ be a formally locally finite type morphism of formal schemes. Then f is formally smooth if and only if $\Omega_{\mathcal{X}/k}$ is finite locally free on $\mathcal{X}$, with rank $\dim \mathcal{O}_{\mathcal{X},x}$ at each closed point $x \in \mathcal{X}$.*

Proof. First assume $\Omega_{\mathcal{X}/k}$ is locally free on $\mathcal{X}$ with rank $\dim \mathcal{O}_{\mathcal{X},x}$ at each closed point of $x \in \mathcal{X}$. This question is local on $\mathcal{X}$ [AJP05, Proposition 2.4.18], so we may assume $\mathcal{X} = \operatorname{Spf} k[T_1, \ldots, T_n][\![x_1, \ldots, x_m]\!]/I$. Write $\mathcal{W} = \operatorname{Spf} k[T_1, \ldots, T_n][\![x_1, \ldots, x_m]\!]$, and let $i\colon \mathcal{X} \to \mathcal{W}$ be the given closed immersion. Let $\mathscr{I}$ be the ideal sheaf of $\mathcal{X}$ on $\mathcal{W}$. There is an exact sequence

$$\mathscr{I}/\mathscr{I}^2 \to i^*\Omega_{\mathcal{W}/k} \to \Omega_{\mathcal{X}/k} \to 0.$$

In our situation, $\mathcal{X} \to \operatorname{Spec} k$ is formally smooth if and only if the preceding exact sequence is also exact on the left [AJP07, Corollary 4.15]. It is enough to check exactness at the stalk of every closed point $x \in \mathcal{X}$. Since $\Omega_{\mathcal{X}/k}$ and $i^*\Omega_{\mathcal{W}/k}$ are locally free on $\mathcal{X}$, with ranks at x given by $n + m$ and $\dim \mathcal{O}_{\mathcal{X},x}$ respectively, it is enough to show that $\mathscr{I}/\mathscr{I}^2$ is free of rank $n + m - \dim \mathcal{O}_{\mathcal{X},x}$ in the stalk at x.

By Lemma A.1, we know that $\mathcal{O}_{\mathcal{X},x}$ is regular. By [Stacks, Lemma 00NR], we know that the stalk $\mathscr{I}_x \subseteq \mathcal{O}_{\mathcal{W},x}$ is generated by a regular sequence of length $n + m - \dim \mathcal{O}_{\mathcal{X},x}$, which gives the claim.

Conversely, if f is formally smooth then $\Omega_{\mathcal{X}/k}$ is finite locally free on $\mathcal{X}$ by Lemma A.2, and the rank claim follows from [AJP09, Proposition 5.9]. □

Proposition A.5. *Let*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\quad f \quad} & \mathcal{Y} \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & \mathcal{S} & \end{array}$$

be a commutative diagram of locally Noetherian formal schemes, where all arrows are formally locally of finite type. Suppose moreover that f is formally smooth, and that the underlying map of topological spaces is surjective. Then g is formally smooth if and only if h is formally smooth.

Proof. The property of being formally smooth is preserved under composition, so if g is formally smooth then h is formally smooth. Suppose h is formally smooth. Since f is faithfully flat and h is flat, we know that g is flat. Combining [AJP09, Corollary 5.5] and

Lemma A.3, we reduce to the case where $\mathcal{S} = \operatorname{Spec} k$ for an algebraically closed field k. We have a short exact sequence

$$0 \to f^*\Omega_{\mathcal{Y}/k} \to \Omega_{\mathcal{X}/k} \to \Omega_{\mathcal{X}/\mathcal{Y}} \to 0$$

since f is formally smooth and formally locally finite type [AJP07, Proposition 4.9]. We know also that $\Omega_{\mathcal{X}/k}$ and $\Omega_{\mathcal{X}/\mathcal{Y}}$ are finite locally free by Lemma A.2. Hence $f^*\Omega_{\mathcal{Y}/k}$ is also finite locally free. Since f is faithfully flat, we conclude that $\Omega_{\mathcal{Y}/k}$ is finite locally free as well. Let $y \in \mathcal{Y}$ be any closed point and let $x \in \mathcal{X}$ be a closed point with $f(x) = y$. Lemma A.4 and the preceding short-exact sequence shows that

$$\begin{aligned} \dim \Omega_{\mathcal{Y}/k} \otimes k(y) &= \dim \Omega_{\mathcal{X}/k} \otimes k(x) - \dim \Omega_{\mathcal{X}/\mathcal{Y}} \otimes k(x) \\ &= \dim \mathcal{O}_{\mathcal{X},x} - (\dim \mathcal{O}_{\mathcal{X}_y,x}) \end{aligned}$$

where $\mathcal{X}_y$ is the fiber of $\mathcal{X}$ over y. Flatness of f implies that $\dim \mathcal{O}_{\mathcal{X}_y,x} = \dim \mathcal{O}_{\mathcal{X},x} - \dim \mathcal{O}_{\mathcal{Y},y}$, so the proposition follows from Lemma A.4. $\square$

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