Grothendieck-Messing deformation theory

Outline

- define p-divisible groups
- construct crystals associated to p-divisible groups
- statement of G-M deformation theory.

Definitions

p prime, S any scheme

$X/S \leftrightarrow$ fppf sheaves on $Sch(S)$.

S-group: commutative fppf sheaf of groups on $Sch(S)$

Def (Grothendieck) p-divisible group is an S-group G satisfying

(1) $p: G \to G$ is epimorphism

(2) $G = \varinjlim_n G(n)$ where $G(n) := \ker(p^n : G \to G)$.

(3) S-groups $G(n)$ representable by finite locally free S-group schemes.

(3)' $G(1)$ ——— $\to G(1) \to G(2) \xrightarrow{p} G(1) \to$

Height rank of $G(1)$ is of the form p^h, where h is a locally constant fn $S \to \mathbb{N}$. When constant, h = height of G.

Equivalent definition (Tate) A p-divisible group is inductive system $(G_n)_{n \in \mathbb{N}}$ of finite locally free S-groups schemes satisfying

(1) $G_n = G_{n+1}(n)$

(2) rank of the fiber of G_n at s is $p^{nh(s)}$ for some locally constant function h on S.

$$G \rightsquigarrow (G(n))_{n\in\mathbb{N}}, \qquad (G_n)_{n\in\mathbb{N}} \to \varinjlim G_n$$

Morphisms are morphisms of fppf sheaves of groups

Tate ; $f_n : G_n \to H_n$ which compatible w/ transition maps

<u>Def</u> $G = (G(n))_{n\in\mathbb{N}}$. Let $G(n)^* := \underline{\mathrm{Hom}}_{Sgrp}(G(n), \mathbb{G}_{m,S})$,

$$p : G(n+1) \twoheadrightarrow G(n)$$

induce

$$p^* : G(n)^* \hookrightarrow G(n+1)^*.$$

The inductive system $(G(n)^*)_{n\in\mathbb{N}}$ wrt the p^*, is a p-divisible group, denoted G^*.

<u>Ex</u> (1) constant S-group

$$(\mathbb{Q}_p/\mathbb{Z}_p)_S := \varinjlim_n \left(\tfrac{1}{p^n}\mathbb{Z}/\mathbb{Z}\right)_S$$

(2) A abelian scheme / S

$$\varinjlim A(n) = \varinjlim \ker(p^n)$$

is a p-divisible group of height $2d$, where $d=$ relative dimension of A/S.

Over formal schemes

$\mathfrak{X}$ adic, locally noetherian formal scheme w/ largest ideal of definition $\mathcal{J}$, then $\mathfrak{X} = \varinjlim_n \mathfrak{X}_n$ where $\mathfrak{X}_n$ = locally spec $\mathcal{O}_{\mathfrak{X}}/\mathcal{J}^{n+1}$, then a p-divisible group over $\mathfrak{X}$ is a system of p-divisible groups G_n over $\mathfrak{X}_n$ such that.

$$G_n = G_{n+1} \times_{\mathfrak{X}_{n+1}} \mathfrak{X}_n$$

Prop Let $\mathfrak{X} = \operatorname{Spf} A$ affine formal scheme, then

$$G \longmapsto (G \bmod J^n)_{n \in \mathbb{N}}$$

is an equivalence of categories $\operatorname{pdiv}(\operatorname{Spec} A) \longleftrightarrow \operatorname{pdiv}(\operatorname{Spf} A)$.

Formal Lie groups

$\operatorname{Inf}^k(G)$ for G any S-group

If G is S-group scheme, $e: S \hookrightarrow G$ cut out by $\mathcal{I}$, $\operatorname{Inf}^k(G)$ is defined by $\mathcal{O}_G/\mathcal{I}^{k+1}$.

$$\overline{G} = \varinjlim \operatorname{Inf}^k(G)$$

Def G is a formal Lie group if

(1) G is formally smooth

(2) $G = \varinjlim_k \mathrm{Inf}^k(G)$
(3) $\forall k$, $\mathrm{Inf}^k(G)$ is representable.

Differentials

- G S-group scheme, then $\omega_G := e^*\Omega^1_{G/S}$, where $e: S \hookrightarrow G$.
- G formal Lie group, then $\omega_G := e^*\Omega^1_{\mathrm{Inf}^k(G)/S}$ for $k \gg 0$.

  ω_G is a f. locally free $\mathcal{O}_S$-module,
  $\dim G :=$ rank of ω_G.

$G \in \mathrm{pdiv}(S)$, where p locally nilpotent on S.
Then

Prop $\bar{G} := \varinjlim_k \mathrm{Inf}^k(G)$ is formal Lie group.

Def $\omega_G := \omega_{\bar{G}}$.
$\dim G :=$ rank ω_G.

Crystals

- Dieudonne crystals for p-divisible groups over perfect fields k of char $p > 0$.
- Generalize to pdiv groups over S w/ p locally nilpotent

Classical case

$W(k) =$ Witt ring, $K_0 = W(k)[\frac{1}{p}]$
$\varphi =$ Frobenius $x \mapsto x^p$ on k

extends to $W(k)$ and K_0

<u>Def</u> crystal is free $W(k)$-module M of finite rank w/ injective φ-linear $F: M \to M$, such that $pM \subseteq FM$.

<u>Def</u> Isocrystal is f. dim'l K_0-vector space N w/ bijective φ-linear map $F: N \to N$.

Morphisms of crystals/isocrystals are $W(k)$ or K_0 linear maps that commute with F.

<u>Def</u> For a p-divisible group G over k,
$D(G) := \mathrm{Hom}(G, CW)$, where CW co-Witt vectors over k.
$\varinjlim W_n$

$E(G) = D(G) \otimes_{W(k)} K_0$.

$F := E(\underbrace{F_{G/k}}_{: G \to G^{(p)}})$.

<u>Thm</u> $G \mapsto D(G)$ is an anti-equivalence of categories $\mathrm{pdiv}(k) \longleftrightarrow$ Dieudonne crystals
Furthermore, rank $D(G) =$ height (G).

<u>General</u>

p locally nilpotent on S,
$G \in \mathrm{pdiv}(S)$
$\mathbb{D}(G)$ is $\mathcal{F}$-crystal on $\mathrm{Crys}(S)$

<u>Def</u> Let $\mathcal{F}$ be a fibered category on (Sch) which

is a stack wrt Zariski topology.

An $\mathcal{F}$-crystal on S is a cartesian section of the fibered category $\mathcal{F} \times_{(Sch)} Crys(X)$, where the map $Crys(X) \to (Sch)$ is

$$(U \hookrightarrow T, \gamma) \longmapsto T.$$

This means for each $(U \hookrightarrow T, \gamma) \in Crys(X)$, we have an object $Q_{(U \hookrightarrow T, \gamma)} \in \mathcal{F}_T$, and for each morphism

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, "f"'] & T \arrow[d, "\bar{f}"] \\ U' \arrow[r, hook] & T' \end{tikzcd}$$

we have an isomorphism

$$u_{\bar{f}} : Q_{(U \hookrightarrow T, \gamma)} \xrightarrow{\sim} \bar{f}^* Q_{(U' \hookrightarrow T', \gamma)}.$$

These satisfy $\bar{f}^*(u_{\bar{g}}) \circ u_{\bar{f}} = u_{\bar{g} \circ \bar{f}}$, where $g : (U' \hookrightarrow T', \gamma') \to (U'' \hookrightarrow T'', \gamma'')$.

- $\mathbb{D}(G)_{(U \hookrightarrow T, \gamma)}$ is f. locally free $\mathcal{O}_T$-module.

So, $G \in pdiv(S_0)$

- $\mathbb{E}(G)$ crystal in fppf groups
- $\overline{\mathbb{E}}(G)$ crystal in formal Lie groups

Def $\mathrm{pdiv}(S_0)'$ full subcategory of $\mathrm{pdiv}(S_0)$ consisting of G_0 w/ property that $\exists$ affine open cover of S_0 by subset U_0, s.t. for any nilpotent immersion $U_0 \hookrightarrow U$, there is a $G_U \in \mathrm{pdiv}(U)$ with $G_U|_{U_0} = G_0|_{U_0}$.

In fact $\mathrm{pdiv}(S_0)' = \mathrm{pdiv}(S_0)$.

Def (vector group)
S scheme, M qcoh $\mathcal{O}_S$-module
$\tilde{M}$:= S-group

$$\Gamma(T, \tilde{M}) = \Gamma(T, M \otimes_{\mathcal{O}_S} \mathcal{O}_T).$$

If M locally free f. rank, then $\tilde{M}$ is vector group.

Def An extension of G by a vector group V is an exact sequence of S groups

$$0 \to V \to E \to G \to 0.$$

It's universal if for any vector group M,

$$\mathrm{Hom}_{\mathcal{O}_S\text{-Mod}}(V, M) \longrightarrow \mathrm{Ext}^1_S(G, M)$$

is an isomorphism.

__Thm__ Assume p locally nilpotent on S, then $\exists$ a universal extension $E(G)$ of G by some vector group $V(G)$. In fact $V(G) = \omega_{G^*}$.

__Prop__ $G, H \in \text{pdiv}(S)$, $u: G \to H$, then $\exists!$ $E(u): E(G) \to E(H)$ s.t.

Induced by $u^*: H^* \to G^*$ $\to$

$$\begin{tikzcd} 0 \arrow[r] & V(G) = \omega_{G^*} \arrow[r] \arrow[d, "V(u)"] & E(G) \arrow[r] \arrow[d, "E(u)"] & G \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & V(H) = \omega_{H^*} \arrow[r] & E(H) \arrow[r] & H \arrow[r] & 0 \end{tikzcd}$$

__Thm__ $S = \text{Spec}\, A$ with $p^N \cdot 1_S = 0$, $S_0 = \text{Spec}(A/I)$ with $I \subseteq A$ an ideal nilpotent divided powers.

Let $G, H \in \text{pdiv}(S)$, $u_0: G_0 \to S_0$ where $G_0 = G|_{S_0}$, $H_0 = H|_{S_0}$.

There exists a unique morphism $E_S(u_0): E(G) \to E(H)$ satisfying the following:

(1) $E_S(u_0)$ lifts $E(u_0)$

(2) Something else

__Corollaries__

1. If K is another pdivisible group and $u_0': H_0 \to K_0$, then $E_S(u_0' \circ u_0) = E_S(u_0') \circ E_S(u_0)$
2. If u_0 is an isomorphism, then $E_S(u_0)$ is an isomorphism.

Def Because fppf groups form a stack, it suffices to define $\mathbb{E}(G_0)$ on $(U_0 \hookrightarrow U, \gamma)$ with U_0 affine.

For this, let G_U be any lift of $G_0|_{U_0}$ to U and define

$$\mathbb{E}(G_0)(U_0 \hookrightarrow U, \gamma) = E(G_U).$$

It's well-defined by corollaries.

For each $f : (U_0 \hookrightarrow U, \gamma) \to (V_0 \hookrightarrow V, \delta)$, we have a canonical isomorphism

$$E(G_U) \xrightarrow{\sim} f^* E(G_V) = E(f^* G_V)$$

($\mathrm{id} : G_U|_{U_0} \to G_V|_{U_0}$, consider $E(\mathrm{id})$).

$\overline{\mathbb{E}(G)}$ and $\mathbb{D}(G)$.

Prop S scheme, p locally nilpotent, $G \in \mathrm{pdiv}(S)$, then $\overline{E(G)} := \varinjlim_k \mathrm{Inf}^k(E(G))$.

Define $\mathrm{Lie}(E(G)) := \mathrm{Lie}(\overline{E(G)}) := (\omega_{\overline{E(G)}})^\vee$.

Prop Apply Lie to universal extension SES, we get

$$0 \to \underset{\mathrm{Lie}(G^*)^\vee}{\underset{\shortparallel}{V(G)}} \to \mathrm{Lie}(E(G)) \to \mathrm{Lie}(G) \to 0.$$

Def

$$\overline{\mathbb{E}(G_0)}_{(U_0 \hookrightarrow U, \gamma)} := \overline{\mathbb{E}(G_0)_{(U_0 \hookrightarrow U, \gamma)}}$$

$\mathbb{D}(G_0)_{(u_0 \hookrightarrow u, \gamma)} := \mathrm{Lie}(\mathbb{E}(G_0)_{(u_0 \hookrightarrow u, \gamma)})$.

rank of $\mathbb{D}(G_0)$ = height of G_0

Relation w/ classical case:

$p \geq 3$

$W_n := W(k)/p^n W(k)$, then we have nilpotent immersions $\mathrm{Spec}\, k \hookrightarrow \mathrm{Spec}\, W_n$ w/ nilpotent divided powers on $pW(k)/p^n W(k)$.

$$\mathbb{D}(G) = \varprojlim_n \mathbb{D}(G^*)_{(\mathrm{Spec}\, k \hookrightarrow \mathrm{Spec}\, W_n)}$$

<u>Deformation theory</u>

Fix $S_0 \hookrightarrow S$ locally nilpotent immersion defined by ideal I with locally nilpotent divided powers. Assume p locally nilpotent on S, $G_0 \in \mathrm{pdiv}(S_0)$.

<u>Def</u> A filtration $\mathrm{Fil}^1 \subseteq \mathbb{D}(G_0)_S := \mathbb{D}(G_0)_{(S_0 \hookrightarrow S)}$ is admissible if Fil^1 is locally a direct factor vector subbundle of $\mathbb{D}(G_0)_S$, and it reduce $V(G_0) \subseteq \mathrm{Lie}(E(G_0))$ on S_0.

<u>Def</u> (Category of admissible pairs)
Objects: (G_0, Fil^1) admissible as above
Morphisms: (u_0, ξ), $u_0 : G_0 \to H_0$, ξ : $\mathrm{Fil}^1(G_0) \to \mathrm{Fil}^1(H_0)$ such that

$$
\begin{tikzcd}
\mathrm{Fil}^1(G_0) \arrow[r, "\xi"] \arrow[d, phantom, "\cap" description] & \mathrm{Fil}^1(H_0) \arrow[d, phantom, "\cap" description] \\
\mathbb{D}(G_0)_S \arrow[r, "{\mathbb{D}(u_0)_S = \mathrm{Lie}(E_S(u_0))}"'] & \mathbb{D}(H_0)_S
\end{tikzcd}
$$

commutes and reduces to

$$
\begin{tikzcd}
V(G_0) \arrow[r, "V(u_0)"] \arrow[d, phantom, "\cap" description] & V(H_0) \arrow[d, phantom, "\cap" description] \\
\mathrm{Lie}(E(G_0)) \arrow[r, "\mathrm{Lie}(E(u_0))"'] & \mathrm{Lie}(E(H_0))
\end{tikzcd}
$$

<u>Theorem</u> (G-M). There's an equivalence of categories

$$\mathrm{pdiv}(S) \simeq \text{category of admissible pairs}$$

$$G \longmapsto \big(G_0 = G|_{S_0},\ V(G) \hookrightarrow \underbrace{\mathbb{D}(G_0)_S}_{=\,\mathrm{Lie}(E(G))}\big)$$