.1 p-Adic Period Domains

Hao Peng, hao_peng@mit.edu

1 Isocrystals

Def. (.1.1.1) (φ-module). Let M be a A-module and $\sigma : A \to A$ is a ring map. Then an additive map $\varphi : M \to M$ is called **σ-semi-linear** iff $\varphi(am) = \sigma(a)\varphi(m)$ for $a \in A$. A **φ-module** over (A, σ) is just an A-module M with a σ-semi-linear φ.

If we define a ring $A_\sigma[\varphi]$ as the free group $A[X]$ modulo the relation $Xa = \sigma(a)X$ and ring relations in A, then it is a ring. Then a φ-module over (A, σ) is equivalent to a left $A_\sigma[\varphi]$-module.

Thus we know that the category of φ-modules is a Grothendieck Abelian category $\Phi\mathcal{M}$ with tensor products, and moreover, the kernel as $A_\sigma[\varphi]$-module is the same as the kernel as a A-module.

Def. (.1.1.2) (Isocrystals). We consider a perfect field k and $K = W(k)[1/p]$, K is equipped with the natural σ lifting the Frobenius. Define an **isocrystal** over K as a f.d. φ-modules V over K with $\sigma = \sigma^a$(.1.1.1), where $a \in \mathbb{Z}\backslash\{0\}$. We don't care about this a much. $\dim_K V$ is called the **height** of this isocrystal. The fact that k is perfect implies that the kernel and image of φ are K-vector subspaces.

Def. (.1.1.3) (Isotypical φ-Modules). A φ-module is called **pure(isotypical) of slope** $\lambda = s/r \in \mathbb{Q}$ if D admits a lattice M on which $p^{-s}\varphi^r$ is a bijection. This is independent of M because λ is independent of M.

Prop. (.1.1.4) (Dieudonné-Manin). If M is a φ-module over $W(k)[\frac{1}{p}]$ where k is a perfect field, then M is a finite sum of modules pure of slopes λ_i. This is called the **isocrystal decomposition** of M.

Proof: We use the $\widetilde{\varphi}$ as in the proof of**??**, we see that M has a decomposition $M_0 \oplus M_{>0}$ by**??**, and $M_0 \neq 0$ by definition. Then we use induction to get the result. □

Def. (.1.1.5). When k is alg.closed, for $\lambda = s/r$, we define a φ-module over $K = W(k)[1/p]$ $E_\lambda = \oplus_{i+0}^{r-1} Ke_i$ that $\varphi(e_i) = e_{i+1}$, and $\varphi(e_{r+1}) = p^s e_0$. In this case, E_λ is irreducible.

Prop. (.1.1.6) (Dieudonné-Manin). If k is alg.closed, then any φ-module over K has a unique decomposition as sums of E_{λ_i}(.1.1.5).

Def. (.1.1.7) (Tate Twist). The Tate object $1(n), n \in \mathbb{Z}$ is the 1-dimensional isocrystal over K_0 that $\varphi = p^n\sigma$, so it is of slope n. And the **Tate twist isocrystal** is tensoring by $1(n)$.

2 Filtered Isocrystals and HN-Formalism

Def. (.1.2.1) (Filtered (φ, N)-Modules). A **filtered φ-module(isocrystals)** (D, φ_D, Fil) over K is a (φ, N)-module $(D, \varphi_D) \in \varphi - Mod_{K_0}$ together with a finite filtration Fil on $D_K = D \otimes_{K_0} K$ in the category of vector spaces over K. The category of filtered φ-modules over K is denoted by $\varphi - FilMod_{K/K_0}$.

Harder-Narasimhan Formalism

Main references are https://arxiv.org/abs/2003.11950.

Def. (.1.2.2) (Harder-Narasimhan Formalism). A **Harder-Narasimhan formalism** consists of

- An exact category $\mathcal{C}$??.
- A function $\deg : Ob(\mathcal{C}) \to \mathbb{Z}$ that is additive w.r.t short exact sequences.
- An exact faithful **generic fiber functor** to an Abelian category $F : \mathcal{C} \to \mathcal{A}$ that induces for each object $F : \mathcal{E} \in \mathcal{C}$ a bijection

$$\{\text{strict objects of } \mathcal{E}\} \cong \{\text{subobjects of } F(\mathcal{E})\}$$

  where a **strict subobject** is an object that can be prolonged to an exact sequence.
- An additive function rank: $\mathcal{A} \to \mathbb{N}$ on $\mathcal{A}$ that $\text{rank}(\mathcal{L}) = 0 \iff \mathcal{L} = 0$, and its composition with F is also called rank.
- If $u : \mathcal{E} \to \mathcal{E}'$ is a morphism in $\mathcal{C}$ that $F(u)$ is an isomorphism, then $\deg(\mathcal{E}) \leq \deg(\mathcal{E}')$ with equality iff u is an isomorphism.

Cor. (.1.2.3).

- We are free to choose the "kernel" for u that $F(u)$ is surjection.
- The subobjects of subobjects are subobjects, by axiom3.

Prop. (.1.2.4) (HN-Formalism on the Category of Filtered Vector Spaces). If L/K is a field extension, there is a category $VectFil_{L/K}$ consisting of (V, Fil) where V is a K-vector space and Fil is a finite filtration on $V \otimes_K L$. It is an exact category by declaring exact sequences be those induce exact sequences on the gradeds.

The generic fiber functor is $VectFil_{L/K} \to Vect_K : (V, Fil) \mapsto V$, and rank is as usual, the **Hodge-Tate degree** is defined to be $t_H((V, Fil)) = \sum i \dim_K gr^i(V \otimes_K L)$. This is a HN-filtration.

Proof: The axioms can be directly checked, notice a filtration W_n of a filtration V_n is a strict object iff $W_k = W_n \cap V_k$. ☐

Def. (.1.2.5) (Slope). In a HN-formalism, the **slope** is defined to be $\text{slope}(E) = \frac{\deg(E)}{\text{rank}(E)}$.

$\mathcal{E}$ is called **semistable of slope** λ iff $\text{slope}(\mathcal{E}) = \lambda$, and $\text{slope}(\mathcal{E}') \leq \lambda$ for any nonzero strict subobject $\mathcal{E}' \subset \mathcal{E}$.

Prop. (.1.2.6) (Semistable Objects). If $f : \mathcal{E} \to \mathcal{F}$ be a map of objects of the same slope λ, then $\ker(f)$ and $\text{Coker}(f)$ are all semistable vector bundles of slope λ, and if $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is exact and $\mathcal{E}', \mathcal{E}''$ are semistable of slope λ, then so does $\mathcal{E}$.

Def. (.1.2.7) (Harder-Narasimhan Filtration). Let $\mathcal{E} \in \mathcal{C}$, a chain of objects $0 \subset \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \ldots \subsetneq \mathcal{E}_m = \mathcal{E}$ is called a **Harder-Narasimhan filtration** iff each quotient $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable of slope λ_i and $\lambda_1 > \lambda_2 > \ldots > \lambda_m$.

Prop. (.1.2.8). Every object $\mathcal{E} \in \mathcal{C}$ has a unique functorial Harder-Narasimhan filtration.

Prop. (.1.2.9) (HN-Formalism for Filtered φ-Modules). The category $\varphi-FilMod_{K/K_0}$(.1.2.1) is a HN-formalism where $\mathcal{A}$ is the Abelian category of φ-modules**??**, the rank is defined as usual and

$$\deg((D,\varphi_D,Fil)) = t_H(D_K,Fil) - t_N(D,\varphi_D)$$

where t_H is the Hodge-Tate degree(.1.2.4) and $t_N = v_p(\det(\varphi_D;D))$. This is a HN-formalism.

Proof: The proof is clear, the same as that of(.1.2.4). □

Fontaine's Rings

Def. (.1.2.10) (Fontaine's Rings). There is a ring B_{dR} which is a complete discrete valued field of residue field characteristic 0 with a G_K-action, it has a natural filtration structure given by the valuation $\mathrm{Fil}^i = t^i B_{dR}^+$, and B_{crys} is a subring of PD-structures of B_{dR}, and B_{crys} is equipped with a Frobenius morphism φ that commutes with the G_K-action.

$B_{dR}^{G_K} = K$, $B_{crys}^{G_K} = K_0$, and there is an embedding $(B_{crys})_{K_0}K \to B_{dR}$.

Def. (.1.2.11) (Admissible Representations). A p-adic representation V of G_K over $\mathbb{Q}_p$ of dimension d is called de Rham iff the B_{dR}-semilinear representation $B_{dR} \otimes_{\mathbb{Q}_p} V$ is trivial.

Similarly, it is called crystalline iff the B_{crys}-semilinear representaion $B_{crys} \otimes_{\mathbb{Q}_p} V$ is trivial.

By the conditions above, if V is crystalline, then it is de Rham.

And for $B = B_{dR}$ or B_{crys} satisfying G_K-regularity, define $D_B(V) = (B \otimes_{\mathbb{Q}_p} V)^{G_K}$ which is a vector space over $F = B^{G_K}$, Fontaine proved that B-admissibility is equivalent to $\dim_F D_B(V) = \dim_{\mathbb{Q}_p}(V)$, equivalently,

$$D_B(V) \otimes_F B \to V \otimes_{\mathbb{Q}_p} B$$

is an isomorphism.

Notice in this way $D_B(V)$ contains filtration structures or φ-action structures inherited from that of B. In particular, if V is crystalline, then $D_{crys}(V)$ is equipped with a φ-module structure and $D_{crys}(V) \otimes_{K_0} K \subset D_{dR}(V)$ equipped with a filtration. Thus this is a filtered isocrystal, and any filtered isocrystal of this form is called **admissible**.

Def. (.1.2.12) (Weakly Admissible is Admissible). If K is a finite field extension of K_0, then weakly admissibility is equivalent to admissibility.

3 Isocrystals with Additional Structures

Main references are [Period Spaces for p-Divisible Groups, Rapoport/Zink] and [Isocrystals with Additional Structures, Kottwitz].

Def. (.1.3.1) (Isocrystals with G-structures over K_0). An **Isocrystal with G-structures over** K_0 is an exact faithful $\otimes$-functor $\mathrm{Rep}_{\mathbb{Q}_p}(G) \to \varphi - \mathrm{Mod}_{K_0}$.

Prop. (.1.3.2) (Associated Isocrystals). Let L be a perfect field of char p and $K_0 = W(L)[\frac{1}{p}]$ and let $b \in G(K_0)$, then to every $\mathbb{Q}_p$-linear representation V of G, we can associate an isocrystal

$$\mathrm{Rep}_{\mathbb{Q}_p}(G) \to \varphi - \mathrm{Mod}_{K_0} : V \mapsto (V \otimes_{\mathbb{Q}_p} K_0, b \circ (\mathrm{id} \otimes \sigma)).$$

this is an isocrystal with G-structures over K_0 associated to b.

If $g \in G(K_0)$ and $b' = gb\sigma(g)^{-1}$, then multiplying by g implies a natural isomorphism between T_b and $T_{b'}$.

Prop. (.1.3.3) (Associated Filtered Isocrystals). Let K be a field extension of K_0, G be an algebraic group over $\mathbb{Q}_p$, and $\mu : \mathbb{G}_{m,K} \to G_K$ be a cocharacter over K, then the associated isocrystal over K_0 upgrades to a filtered isocrystal over K(.1.2.1),

$$\mathrm{Rep}_{\mathbb{Q}_p}(G) \to \varphi - FilMod_K\mathcal{I} : V \mapsto (V \otimes_{\mathbb{Q}_p} K_0, b \circ (\mathrm{id} \otimes \sigma), \mathrm{Fil}^{\bullet}_{\mu}),$$

where the filtration comes from μ by weight-filtrations($\mathbb{G}_m$ is diagonalizable**??**).

Def. (.1.3.4) (Admissible Pair). Let G be a reductive group, then a pair (μ, b) in(.1.3.3) is called a **(weakly)admissible pair** if for any $\mathbb{Q}_p$-representation of G, the filtered isocrystal $\mathcal{I}(V)$ is (weakly)admissible**??**(.1.2.11).

It suffices to check this condition for any faithful representation V.

Proof: This is because for a faithful representation V, any $\mathbb{Q}_p$-representation appears as a direct summand of $V^{\otimes n} \otimes \widehat{V}^{\otimes m}$**??**. Then the assertion follows from the fact direct summands and tensor products of (weakly)admissible filtered isocrystals is (weakly)admissible. □

Prop. (.1.3.5) (Slope Morphism). Let $\mathbb{D} = \mathrm{Spec}\, \mathbb{Q}_p[\{T^{1/k}\}_{k \in \mathbb{Z}}] = \mathbb{D}(\mathbb{Q})_{\mathbb{Q}_p}$ be the pro-algebraic torus over $\mathbb{Q}_p$ with character group $\mathbb{Q}$, and $b \in G(K_0)$, then there is a morphism $\nu : D_{K_0} \to G_{K_0}$, called the **slope morphism** associated to b, which is defined as follows:

For any f.d. representation $\rho : G \to GL(V)$, there is an associated isocrystal defined in(.1.3.2), then there is a morphism $\nu_\rho \in \mathrm{Hom}_L(\mathbb{D}, GL(V))$ that $\mathbb{D}$ acts on the isotypical component V_λ of V by the character $\lambda \in \mathbb{Q} = X^*(\mathbb{D})$. Then for any $x \in G(R)$, the mapping $\rho \to \mu_\rho$ gives an automorphism of the standard fiber functor on $\mathrm{Rep}(G)$, so by Tannakian duality corresponds to a unique element $y \in G(R)$ that $\rho(y) = \nu_\rho(x)$ for any ρ. The homomorphism $x \mapsto y$ is functorial in R and thus defines an element $\nu \in \mathrm{Hom}_L(\mathbb{D}, G)$.

Notice the group $\mathbb{Q}^*$ acts on $\mathbb{D}$, and for $s \in \mathbb{Q}^*$ and $v \in \mathrm{Hom}(\mathbb{D}, G)$, denote by sv the composite $\mathbb{D} \xrightarrow{s} \mathbb{D} \xrightarrow{v} G$, and $D \to \mathbb{G}_m$ the natural morphism, then for any v, there is a suitable s that sv factors through a morphism also denoted by $s\nu : \mathbb{G}_{m,K_0} \to G_{K_0}$, as G is algebraic.

Prop. (.1.3.6) (Characterizing Slope Morphism). The slope morphism can be characterized intrinsically to be the unique morphism $\nu \in \mathrm{Hom}_L(\mathbb{D}, G)$ that there exists some $s > 0, c \in G(L)$ that

- $s\mu \in \mathrm{Hom}_L(\mathbb{G}_m, G)$,
- $c(s\mu)c^{-1}$ is defined over $\mathbb{Q}_{p^s}$.
- $c(b\sigma)^n c^{-1} = c(n\nu)(p)c^{-1}\sigma^n$.

Proof: Cf.[Kottwitz, P13]. □

Cor. (.1.3.7) (Conjugate of Slope Morphism). Now we can define the σ-conjugate of the slope morphism ν, and we have the identity

$$b\nu^\sigma b^{-1} = \nu$$

To check this, replace ν by some $s\nu$ to assume ν factors through $\mathbb{G}_m$, then it suffices to show for any $a \in K_0^*$,

$$b\sigma(s\nu(\sigma^{-1}(a))) = s\nu(a)b\sigma$$

It suffices to check for any G-representation $\mathbb{Q}$, and it is true as $(V \otimes_{\mathbb{Q}_p} K_0, b\sigma)$ is a φ-module for σ and $s\nu(a)$ acts on the isotypical part of slope r by a^r.

More generally enerally, any $g \in G(L)$ commuting with $b\sigma$ also commutes with $\varphi(a), a \in K_0^*$, as it preserves the isotypical decomposition for any isocrystal, and on the isotypical component V_λ, the a acts by a^r.

Def. (.1.3.8) (Descent Conditions). A σ-conjugacy class $\bar{b}$ in $G(\overline{K})$ is called a descent if there is some $s \geq 0$ and some $b \in \bar{b}$ that $s\nu$ factors through $D \to \mathbb{G}_m$ and

$$(b\sigma)^s = s\nu(p)\sigma^s$$

as an identity in $G(K_0) \rtimes \langle\sigma\rangle$.

Prop. (.1.3.9). If G is connected and L is alg.closed, then any σ-conjugacy class is descent(.1.3.8).

Proof: Cf.[Kottwitz]. □

Prop. (.1.3.10). Let $\bar{b}$ be a descent and b satisfies the descent condition for s in(.1.3.8), then if $\mathbb{Q}_{p^s} = W(\mathbb{F}_{p^s})[\frac{1}{p}]$, $b \in G(\mathbb{Q}_{p^s})$ and ν is defined over $\mathbb{Q}_{p^s}$.

Proof: Set $b_s = b\sigma(b)\ldots\sigma^{s-1}(b)$, then iterating(.1.3.7), $b_s\nu^{\sigma^s}b_s^{-1} = \nu$. And we have $b_s = s\nu(p)$, so $\nu^{\sigma^s} = \nu$, so ν is defined over $\mathbb{Q}_{p^s}$.

To show the first assertion, notice $(b\sigma)(b\sigma)^s = (b\sigma)^s(b\sigma)$ shows

$$s\nu(p)\sigma^s b\sigma = b\sigma s\nu(p)\sigma^s = s\nu(p)b\sigma^{s+1}(.1.3.7).$$

and then $b\sigma^s = \sigma^s b$. □

Cor. (.1.3.11). If $b_1, b_2 \in \bar{b}$ are descent w.r.t the same s, then they are conjugate w.r.t. $G(K_0 \cap \mathbb{Q}_{p^s})$.

In particular, for any descent $b \in G(\mathbb{Q}_{p^s})$ and any $\mathbb{Q}_p$-representation V of G, the induced isocrystal is defined over the field $K_s = W(\mathbb{F}_{p^s} \cap L)[\frac{1}{p}]$, and it only depends on $\bar{b}$ up to isomorphism.

Proof: Suppose $b_2 = gb_1\sigma(g)^{-1}$, then $\nu_2 = g\nu_1 g^{-1}$, and the descent equations are

$$(b_1\sigma)^2 = s\nu_1(p)\sigma^s, \quad g(b_1\sigma)^s g^{-1} = gs\nu_1(p)g^{-1}\sigma^s.$$

Comparing these two, g commutes with σ^s, so $g \in G(K_0 \cap \mathbb{Q}_{p^s})$. □

Prop. (.1.3.12). Let $b \in G(K_0)$, then the following functor on the category of $\mathbb{Q}_p$-algebras is representable by a smooth affine group scheme:

$$J(R) = \{g \in G(R \otimes_{\mathbb{Q}_p} K_0) | g(b\sigma) = (b\sigma)g\}.$$

Moreover, if $b \in G(W(L')[\frac{1}{p}])$ where L' is an alg.closed subfield of L, and J' be the corresponding functor defined with L', then the canonical morphism $J' \to J$ is an isomorphism.

Proof: Choose an embedding $G \subset GL(V, \mathbb{Q}_p)$, consider the functor:

$$F(R) = \{g \in \mathrm{End}(V_{K_0}) \otimes R | g = b\sigma(g)b^{-1}\},$$

then it is representable by an affine space by the lemma(.1.3.13) applied to the σ-linear map $g \mapsto b\sigma(g)b^{-1}$.

More precisely, there is a f.d. $\mathbb{Q}_p$-vector space $W \subset \mathrm{End}(V_{K_0})$ that $F(R) = W \otimes_{\mathbb{Q}_p} R$. Choose a basis A_i of W, then $J(R)$ is just the subfunctor of $r_i \in R$ that $f_k(\sum r_i A_i) = 0$, and $\det(\sum r_i A_i) \neq 0$. Taking a basis of K_0 over $\mathbb{Q}_p$, then these are polynomials with coefficients in $\mathbb{Q}_p$. It is automatically smooth by Cartier's theorem**??**.

The last assertion follows from the proof of(.1.3.13). □

Lemma (.1.3.13). Let N be a f.d. isocrystal over $K_0 = W(L)[\frac{1}{p}]$ w.r.t. σ^s for some $s \neq 0$, then the following functor on the category of $\mathbb{Q}_p$-algebras

$$F(R) = \{n \in N \otimes_{\mathbb{Q}_p} R | \varphi(n) = n\}$$

is representable by an affine space over $\mathbb{Q}_p$.

Proof: $F(R)$ is just $N^\varphi \otimes_{\mathbb{Q}_p} R$, so it suffices to show $\dim_{\mathbb{Q}_p} N^\varphi < \infty$. Firstly assume that L is alg.closed, then this is a consequence of Dieudonné-Manin classification(.1.1.6). This functor F doesn't depend on L once L reaches its alg.closure: if L is alg.closed and L' is a field extension, then the corresponding functor F' defined by $N \otimes_{W(L)[\frac{1}{p}]} W(L')[\frac{1}{p}]$ coincide with F.(This is also by Dieudonné-Manin classification.) □

Cor. (.1.3.14). Assume b satisfies a descent condition for s(.1.3.8), then J is a $\mathbb{Q}_{p^s}/\mathbb{Q}_p$-inner form of the centralizer $G_{s\nu(p)}$(.1.3.10).

Proof: The descent equation shows $b_s = s\nu(p)$, so the adjoint $b_{ad} : g \mapsto (b\sigma)g(b\sigma)^{-1} = b\sigma(g)b^{-1}$ defines an element in $H^1(G(\mathbb{Q}_{p^s}/\mathbb{Q}_p), \mathrm{Aut}(G_{s\nu(p)}(\mathbb{Q}_{p^s})))$, because

$$\sigma^k b_{ad} : g \mapsto \sigma(b\sigma^{-1}(g))b^{-1})) = \sigma^k(b)\sigma(g)\sigma^k(b)^{-1}.$$

so

$$b_{ad} \circ \sigma(b_{ad}) \circ \ldots \circ \sigma^{s-1}(b_{ad}) : g \mapsto b_s g b_s^{-1} = s\nu(p)g(s\nu(p))^{-1} = g.$$

So it defines an inner form, which is just

$$J'(R) = G_{s\nu(p)}(\mathbb{Q}_{p^s}))^{b_{ad}\sigma} = \{g \in G_{s\nu(p)}(R \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^s}) | g(b\sigma) = (b\sigma)g\}$$

Now it suffices to show $J'(R)$ is just $J(R)$ defined in(.1.3.12). For this, notice any $g \in J(R)$ commutes with $b\sigma$ thus commutes with $s\nu(p)$ by(.1.3.7), and the descent condition $(b\sigma)^n = s\nu(p)\sigma^n$ shows it commutes with σ^n, so $g \in J'(R)$. □

Prop. (.1.3.15). Let G be a connected reductive group and L be alg.closed, then the following are equivalent for $b \in G(K_0)$:

- The slope morphism ν factors through the center of G.
- b is σ-conjugate to an element in $T(K_0)$ where T is an elliptic maximal torus of G.
- The algebraic group J of(.1.3.12) is an inner form on G.

In this case, b and its conjugacy class $\overline{b}$ are called **basic**.

Proof: Cf.[Kottwitz]. □

Prop. (.1.3.16) (Conjugacy Classes and Base Change). Let b_1, b_2 be two elements of $G(W(L)[\frac{1}{p}])$, then the functor

$$J(R) = \{g \in G(R \otimes_{\mathbb{Q}_p} W(L)[\frac{1}{p}]) | g(b_1\sigma) = (b_2\sigma)g\}$$

is representable by a smooth affine scheme over $\mathbb{Q}_p$.

Assume $b_1, b_2 \in G(W(L')[\frac{1}{p}])$ where L' where L' is an alg.closed field of L, and J' the corresponding functor, then $J' \to J$ is an isomorphism. In particular, the map from the set of σ-conjugacy classes in $G(W(L')[\frac{1}{p}]$ to the set of σ-conjugacy classes in $G(W(L)[\frac{1}{p}])$ is injective, and it is surjective iff L is also alg.closed and G is connected.

Proof: The surjectivity follows from the fact that every conjugacy class is descent(.1.3.9), and those descent elements are in $G(\mathbb{Q}_s)$ for some $s \geq 0$(.1.3.10), so in $G(W(L')[\frac{1}{p}])$. □

4 Period Domain

Def. (.1.4.1) (Associated Partial Flag Variety). Let G be an algebraic group over $\mathbb{Q}_p$ and $\mu : \mathbb{G}_m \to G$ is a conjugacy class of cocharacters defined over a finite extension field $E/\mathbb{Q}_p$**??**, then there is associated a faithful $\otimes$-functor

$$\mathrm{Rep}_{\mathbb{Q}_p} \to \mathbb{Z}\text{-graded } R\text{-vector spaces} \to \text{filtered } E\text{-spaces}$$

Now call two cocharacters equivalent if their associated functor are isomorphic. Consider the functor

$$R \mapsto \{\text{the equivalence classes in the conjugacy class of } \mu_R \text{ under } G(R)\}$$

in the category of E-algebras, and also consider the closed algebraic subgroup $P(\mu) \subset G$ over E:

$$P(\mu)(R) = \{g \in G(R) | g\mu_R g^{-1} \text{ is equivalent to } \mu_R\}$$

then the functor above is representable by the homogenous variety $\mathcal{F} = G_E/P(\mu)$ defined over E.

Prop. (.1.4.2). $\mathcal{F}$ is a projective variety.

Proof: If V is a faithful representation in $\mathrm{Rep}_{\mathbb{Q}_p}(G)$, we denote $Flag(V)$ the partial flag variety over $\mathbb{Q}_p$ which associates to any $\mathbb{Q}_p$-algebra R the filtration $\mathrm{Fil}^\bullet$ of $V \otimes_{\mathbb{Q}_p} R$ s.t. $\mathrm{gr}^i(R)$ are direct summands and $\mathrm{rk}\,\mathrm{Fil}^i = \dim_E \mathrm{Fil}^i_\mu(V_E)$. Then $Flag(V)$ is a projective variety, by classical results, and there is a closed immersion

$$\mathcal{F} \hookrightarrow Flag(V)_E$$

because the isocrystal on other representations are determined by this faithful representation. □

Def. (.1.4.3) (p-adic Period Space). Let $\breve{E} = EK_0(\overline{F}_p)^\wedge$ be the completion of the maximal unramified extension of E, then there is a rigid-analytic structure on $\breve{\mathcal{F}} = \mathcal{F}_{\breve{E}}$. define the **$p$-adic period space** $(\breve{\mathcal{F}}^{wa}_b)^{rig} \subset \breve{\mathcal{F}}^{rig}$ associated to $(G, b\{\mu\})$ the set of points ξ conjugate to μ that (ξ, b) is weakly admissible.

Let J_b be the algebraic group associated to b as in(.1.3.12), then $J_b(\mathbb{Q}_p) \subset G(K_0)$ acts on $\breve{\mathcal{F}}^{rig}$, and it preserves the set $(\breve{\mathcal{F}}^{wa}_b)^{rig}$.

$(\breve{\mathcal{F}}^{wa}_b)^{rig}$ has a natural structure of an admissible open subset of $\breve{\mathcal{F}}^{rig}$. if $b' = gb\sigma(g)^{-1}$, then $\mu \mapsto g^{-1}\mu g$ induces an isomorphism from $(\breve{\mathcal{F}}^{wa}_b)^{rig}$ to $(\breve{\mathcal{F}}^{wa}_{b'})^{rig}$. Moreover, if b satisfies descent condition w.r.t. $s > 0$, then this admissible open subset is defined over $E.\mathbb{Q}_{p^s}$.

Proof: Cf.[Rapoport Zink, P26]. □

5 Algebraic Groups of EL/PEL Types

Def. (.1.5.1) (Algebraic Groups of EL/PEL Types). Let F be a finite étale algebra over $\mathbb{Q}_p$, B a finite central algebra over F, and V is a f.g. B-module.

An **algebraic group of EL type** over $\mathbb{Q}_p$ is an algebraic group of the form $GL_B(V)$. They are related to the classification of p-divisible groups with an endomorphism and level structures.

Let $(-,-)$ be a non-degenerate alternating $\mathbb{Q}_p$-bilinear form on V together with a formal involution $*$ on B that

$$(bv, w) = (v, b^* w).$$

Let F_0 be the field of elements of F fixed by $*$.

An **algebraic group of PEL type** over $\mathbb{Q}_p$ is an algebraic group over $\mathbb{Q}_p$ given by

$$G(R) = \{g \in GL_B(V \otimes_{\mathbb{Q}_p} R) | \exists c \in X(G), \quad (gv, gw) = c(g)(v, w), \quad \forall v, w\}$$

Prop. (.1.5.2) (Setups). If G is an algebraic group of EL/PEL type, $K_0 = W(\overline{\mathbb{F}_p})[\frac{1}{p}]$, $b \in G(K_0)$, then we associate to b and the natural representation of G on V the isocrystal

$$(N(V), \Phi) = (V \otimes_{\mathbb{Q}_p} K_0, b(1 \otimes \sigma)).$$

This isocrystal is equipped with an action of B, and in the PEL case an alternating bilinear form

$$\psi : N(V) \otimes N(V) \to 1(n).$$

where $n = v_p(c(b))$. In fact, we can find some unit u that $c(b) = p^n u \sigma(u)^{-1}$, then the pairing is defined as

$$\psi(v, v') = u^{-1}(v, v'),$$

any other choices of u multiplies ψ by an element in $\mathbb{Z}_p^*$.

We will fix in addition a conjugacy class of cocharacters $\mu : \mathbb{G}_m \to G$ defined over a field E, and the associated homogenous algebraic variety $\mathcal{F}$ defined over E of filtrations(.1.4.1). $\mathcal{F}$ is equipped with a B-action, as $G \in GL_B(V)$.

Notice in the PEL case, these filtrations satisfy $\mathcal{F}^i = (\mathcal{F}^{m-i+1})^\perp$, where $m = c\circ\mu \in \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$. This is due to the fact $(kv, kw) = k^m(v, w)$ and the fact the pairing is non-degenerate.

Prop. (.1.5.3) (Shimura Field). Fix a conjugacy class of cocharacters $\{\mu\}$ defined over E and $\mu_0 \in \{\mu\}$, its corresponding filtration $\mathcal{F}_0^\bullet$, The field E in(.1.5.2) can be described as the field of definition of the isomorphism class of $\mathcal{F}_0^\bullet$ as a B-invariant filtration, or equivalently as the finite extension of $\mathbb{Q}_P$ generated by the traces

$$\mathrm{tr}(d; \mathrm{gr}^i_{\mathbb{F}_0}(V \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p})), d \in B, i \in \mathbb{Z}.$$

And the filtration $\mathcal{F}$ is described as the functor that for any E-algebra R, $\mathcal{F}(R)$ is the set of filtrations $\mathcal{F}^\bullet$ of $V \otimes_{\mathbb{Q}_p} R$ by R-modules that are direct summands that

$$\mathrm{tr}(d; \mathrm{gr}^i_{\mathcal{F}}(V \otimes_{\mathbb{Q}_p} R)) = \mathrm{tr}(d; \mathrm{gr}^i_{\mathbb{F}_0}(V \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p})).$$

and moreover in the PEL case satisfies $\mathcal{F}^i = (\mathcal{F}^{m-i+1})^\perp$.

Proof: 1: The field of definition E of the conjugacy class $\{\mu\}$ is determined by Tannakian duality, so it suffices to check over which field these two filtrations are isomorphic as G-filtrations, but G is just the group fixing the B-module structure, so it suffices to show they are equivalent as B-modules, which is then determined by the traces, by**??**.

2: It suffices to show $\mathcal{F}$ is a homogenous space under G. We restrict to the PEL case, the EL case is simpler. After base change from $\mathbb{Q}_p$ to $\overline{\mathbb{Q}_p}$, the data decomposes to the following types:

- $(A) : B = \mathrm{End}(W) \times \mathrm{End}(W^\vee)$ where W is a f.d. $\overline{\mathbb{Q}_p}$-vector space and $(u, v)^* = (v^t, u^t)$. And $V = W \otimes V' \bigoplus W^\vee \otimes V'^\vee$ where the pairing is natural and makes the sum orthogonal.

$$G = \{(1 \otimes g, c \cdot (1 \otimes g^{-t}) | g \in GL(V'), c \in X(G)\}$$

- $(C) : B = \mathrm{End}(W)$ where W is a f.d. $\overline{\mathbb{Q}_p}$-vector space equipped with a symmetric bilinear form $(-,-)_W$ and $*$ is the transposition w.r.t it.
  And $V = W \otimes V'$ where V' is equipped with an alternating form $(-,-)_{V'}$ that $(-,-)_V = (-,-)_W \otimes (-,-)_{V'}$.

$$G = \{cg | g \in \mathrm{Sp}(V'), c \in X(G)\}$$

- (BD): As in (C), except that $(-,-)_W$ is skew-symmetric and $(-,-)_{V'}$ is symmetric.

$$G = \{cg | g \in SO(V'), c \in C(G)\}$$

Under this decomposition, the functor $\mathcal{F}$ in the proposition is represented by products of partial flags of V:

- $(A) : \mathcal{F}^i = W \otimes (\mathcal{F}')^i \bigoplus W^\vee \otimes ((\mathcal{F}')^{m+1-i})^\perp$ and the correspondence $\mathcal{F}^\bullet \mapsto (\mathcal{F}')\bullet$ identifies $\mathcal{F}$ with the partial flag variety of V' with fixed dimensions $\dim((\mathcal{F}')^i)$.
- $(B, CD) : \mathcal{F}^\bullet = W \otimes (\mathcal{F}')^\bullet$ and $\mathcal{F}$ is identified with the partial flag variety of V' of fixed dimensions $\dim((\mathcal{F}')^i)$ and $(\mathcal{F}')^i = ((\mathcal{F}')^{m+1-i})^\perp$.
  The (A) case G clearly acts transitively on $\mathcal{F}$, and the (B, CD) case $(\mathcal{F}')^i$ is isotropic for $i \geq (m+1)/2$, and it determines all other components, so G acts transitively, by Witt's theorem**??**.

The reason is**??** and the fact representations of B is semisimple, then contemplating on the pairing condition. □

Prop. (.1.5.4) (Examples of PEL Type). Let $B = D$ be the quaternion algebra over $\mathbb{Q}_p$ and $*$ be the involution, i.e.

$$D = \mathbb{Q}_{p^2}[\Pi], \quad \Pi^2 = p, \quad \Pi a = \sigma(a)\Pi$$

and

$$a^* = \sigma(a), a \in \mathbb{Q}_{p^2}, \quad , \Pi^* = \Pi.$$

Let (V, ι) be a free D-module of rank n with a non-degenerate bilinear form satisfying the conditions in(.1.5.1). Then G is a non-trivial inner form of the group GSp_{2n} of symmetric similitudes:

Firstly $\mathbb{Q}_{p^2} \otimes K_0 \cong K_0 \oplus K_0$, then $\mathbb{Q}_{p^s}$ acts on $K_0 \oplus K_0$ by $a(x, y) = ax, \sigma(a)y)$. As V is a $\mathbb{Q}_{p^2}$-vector space, there is a decomposition

$$V = V_0 \oplus V_1$$

where $\mathbb{Q}_{p^2}$ acts on V_i by $a(v) = v.\sigma^i(a)$, then G_{K_0} is just GSp_{2n,K_0}, and $G \neq GSp_{2n}$ as the Galois action σ on $\mathbb{Q}_{p^2} \otimes K_0$ and $K_0 \cong K_0 \oplus K_0$ are different.

Take $b \in G(K_0)$ the element with $c(b) = p$ and the corresponding isocrystal (N, Φ) is isotypical of slope $1/2$. N decomposes as $N_0 \oplus N_1$. Notice now Π and $\Phi = b\sigma$ interchanges N_i, and $\Pi\Phi = \Phi\Pi$. Also N_i is isotropic: For $v, w \in N_i, a \in \mathbb{Q}_{p^2}$,

$$a(v, w) = (av, w) = (\iota(\sigma^i(a))v, w) = (v, \iota(\sigma^{i+1}(a))w) = (v, \sigma(a)w) = \sigma(a)(v, w)$$

so $(v, w) = 0$.

We can define a new non-degenerate alternating form

$$\langle -, - \rangle : N_0 \times N_0 \to K_0 : \langle v, v' \rangle = (v, \Pi v')$$

and also a σ-linear endomorphism of N_0: $\Phi_0 = \Pi^{-1} \circ \Phi|_{N_0}$. From the condition, $v_p(\det \Phi_0) = 0$, and Φ has all the slopes 0. Also $\langle \Phi_0 v, \Phi_0 w \rangle = \sigma(\langle v, w \rangle)$, as

$$\langle \Phi_0 v, \Phi_0 w \rangle = (\Pi^{-1}\Phi v, \Phi w) = (\Pi^{-1} b\sigma v, b\sigma w) = \sigma(v, \Pi w) = \sigma(\langle v, w \rangle).$$

so this alternating form is defined over $\mathbb{Q}_p$, denoted by $(V_0, \langle -, - \rangle)$, and Φ_0 corresponds to σ. Then $J_b = GSp(V_0, \langle -, - \rangle)$.

Next we consider

$$(0) = \mathcal{F}_0^2 \subset \mathcal{F}_0^1 \subset \mathcal{F}_0^0 = V \otimes \overline{\mathbb{Q}_p}$$

be a filtration where $\mathcal{F}_0^1$ be a D-invariant Lagrangian subspace. This corresponds to a cocharacter $\mu \to G$, and $\mathcal{F}$ is just the $\mathbb{Q}_{p^2}$ variety of D-invariant Lagrangian subspaces of $V_{\mathbb{Q}_{p^2}}$. By(.1.5.3), the Shimura field is $\mathbb{Q}_p$.

Let $\mathcal{F} \subset \mathcal{F}(K)$ where K/K_0 is a field extension, then

$$\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$$

where $\mathcal{F}_i \in N_0 \otimes_{K_0} K$, as $\mathcal{F}$ is Π-invariant. Now $\mathcal{F}_0$ is also a Lagrangian subspace of $(V_0, \langle -, - \rangle)$. $\mathcal{F}(K)$ identifies the K-points of the Grassmannian of Lagrangian subspaces of $(V_0, \langle -, - \rangle)$.

Cor. (.1.5.5). Under the above identification, the subset $\mathcal{F}^{wa}(K)$ of the Grassmannian of Lagrangian spaces $\mathcal{F}$ of $(V_0 \otimes K, \langle -, - \rangle)$ is characterized by $\mathcal{F}$ satisfying the the following conditions:

For all totally isotropic subspaces $W_0 \subset V_0$, we have $\dim_K \mathcal{F} \cap (W_0 \otimes K) \leq 1/2 \dim W_0$.

Proof: It's clear $\mu(N, \Phi, \mathcal{F}) = 0$, so weakly-admissibility is equivalent to semi-stability. The uniqueness of the HN-filtration of $\mathcal{F}$ implies its D-invariance, thus semi-stability is equivalent to the fact that for any subspace $P \subset N$ stable under Φ and D-action, we have

$$\dim_K(\mathcal{F} \cap (P \otimes_{K_0} K)) \leq v_p(\det(\Phi; P)).$$

Now Φ is isotypical with slope $1/2$, $v_p(\det(\Phi; P)) = \frac{1}{2} \dim P$, and the D-invariance of P is equivalent to $P = P_0 \oplus P_1$ and the Φ-invariance of P is equivalent to the Φ_0-invariance of P_0, i.e. P_0 is a $\mathbb{Q}_p$-rational subspace $W_0 \subset V_0$.

Finally we show it suffices to check for totally isotropic subspaces: Let W_0' be the radical of W_0, then there is a non-singular alternating form on W_0/W_0', then the image of $\mathcal{F}_0' \cap (P \otimes_{K_0} K)$ in this quotient is a totally isotropic space, thus has dimension$\leq \frac{1}{2} \dim(W_0/W_0')$. then it suffices to check the condition for W_0'. □