Spherical varieties and $L$-functions

Zhiyu Zhang

MIT

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Outline

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   - Motivation: images of Langlands transfer
   - Motivation: period integrals and $L$-functions

2 Spherical varieties
   - Dual group
   - Classification
   - Degeneration

3 More Ichino-Ikeda
   - Generalized global Ichino-Ikeda conjectures
Part 1: Origins
Local story (main):
1. $F = \mathbb{Q}_p$ or $F = \mathbb{F}_q((t))$ is a local field.
2. $G$ is a (split) reductive group over $F$.
3. $\pi$ is an irreducible (unitary) smooth $\mathbb{C}$-coefficient representation of $G(F)$.
4. $W_F$ is the Weil-Deligne group of $F$.

Global story:
1. $k = \mathbb{Q}$ or $k = \mathbb{F}_q(C)$ is a global field.
2. Still denote by $G$ a (split) reductive group over $k$.
Branching laws

$\pi$ an irr rep of $G(F)$, $H \subseteq G$ a nice ("spherical") subgroup.
A central problem in representation theory: when $\text{Hom}_H(\pi, 1) \neq 0$?
How to produce elements in $\text{Hom}_H(\pi, 1)$? $\dim_\mathbb{C}\text{Hom}_H(\pi, 1) =$? 
Global analog for automorphic representations?
Frobenius reciprocity

$$\text{Hom}_H(\pi, 1) \neq 0 \iff \pi \hookrightarrow C^\infty(G(F)/H(F)).$$
This motivates the spectral study of function spaces on $F$-points of the spherical variety $X = G/H$.
A subtly is that $G(F)/H(F) \neq (G/H)(F)$, we will ignore the nature of inner forms and $L$-packets in this talk.
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**Today:**

- Why spherical \( X \)? Langlands transfer, period integrals and \( L \)-functions...
- How to think about spherical varieties? Examples, classifications...
- What is Ichino-Ikeda type conjecture? Relations between period integral and central \( L \)-values, relation between local and global program...

We also ignore convergence issues or derived branching laws, so things will be supercuspidal (local) and cuspidal (global).
Motivation: images of Langlands transfer

Local Langlands correspondence and functoriality (informal)

\[ \{ \text{irr smooth } \mathbb{C}\text{-reps of } G(F) \} \xrightarrow{\text{finite-to-one}} \{ \text{Galois reps } \phi : W_F \to G^\vee(\mathbb{C}) \} \]

Given good \( f : G_1^\vee \to G_2^\vee \), composition by \( f \) on RHS gives a transfer map from irr reps of \( G_1 \) to irr reps of \( G_2 \) on LHS.

Example: quadratic base change, parabolic induction.

Question (local/global)

Choose a map \( G_X^\vee \to G^\vee \). For a rep \( \pi \) of \( G(F) \), when does \( \phi_\pi : W_F \to G^\vee(\mathbb{C}) \) factor through \( G_X^\vee \)? For example, when is \( \pi \) a transfer from \( G_1 \)?
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Lots of examples $\rightsquigarrow$ we can detect this by

- (global) poles or nonvanishing of certain $L$-function at $s = s_0$ (center or nearly center points);
- (global) nonvanishing of certain automorphic period integrals;
- (local) certain branching laws $\text{Hom}_H(\pi, 1) \neq 0$.

So another question: relation between period integrals and $L$-functions in general?
Reminder on $L$-functions

$$
\zeta(s) = \prod_p (1 - p^{-s})^{-1} = \sum_n \frac{1}{n^s}.
$$

Given a rep $\pi$ of $G(F')$, and $\rho : G^\vee \to \text{GL}(V)$, the associated local $L$-function ($s \in \mathbb{C}$) is

$$
L(\pi, \rho, s) = \det(1 - q_F^{-s} \rho \circ \phi_\pi(\text{Frob}_F)|_{V_I})^{-1} \in \mathbb{C}[q^s, q^{-s}].
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\]

This is defined unconditionally for unramified $\pi$ by Satake isomorphism. Given a rep $\pi$ of $[G]$, and $\rho : G^\vee \to \text{GL}(V)$, the associated (incomplete) global $L$-function is

\[
L(\pi, \rho, s) = \prod_v L(\pi_v, \rho_v, s)
\]

where the product is over finite places $v$ of $k$ where $\pi_v$ is unramified.
Let $H \subseteq G$ be a "nice" subgroup. The "niceness" is encoded in $X = G/H$, e.g. $X$ is affine iff $H$ is reductive. Below we ignore important convergence issues.

### Automorphic period integrals

(general) For $\phi \in \pi$ on $[G]$, $P_X(\phi) := \int_H \phi(h)dh$. 

Twisted version: insert "small" functions e.g. a character of $H$, or a small kernel. Relations $L$-functions: $\int_H \phi(h)dh = (\ast) L(\cdot, \cdot, s_0)$. 

Branching laws: note $P_X \in \text{Hom}_H(\pi, 1)$, so $P_X \neq 0$ implies $\text{Hom}_H(\pi, 1) \neq 0$, i.e. $\pi$ is $H$-distinguished.

Langlands transfer: functoriality shall also be realized by integration along certain kernel functions (geometrization).
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Local period integrals?

In practice, people study or construct $L$-functions by relating it to some (period) integrals e.g to show analytic continuation. See Tate thesis. \( \rightsquigarrow \) how about local decomposition of global period integrals? 

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In practice, people study or construct $L$-functions by relating it to some (period) integrals e.g to show analytic continuation. See Tate thesis. → how about local decomposition of global period integrals?

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Ichino-Ikeda type conjecture: global $|P_H|^2$ can be decomposed into local pairings $(v_1, v_2) \mapsto \int_H \langle h.v_1, v_2 \rangle \, dh$, $v_1 \in \pi, v_2 \in \pi^\vee$, after some important normalizations. This will relate central $L$-values to period integrals, in a precise way.
Examples

We give examples for previous two questions.

- Dirichlet $L$-function $L(\chi, s)$ has a pole at $s_0 = 1$ iff $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ is trivial i.e its Langlands parameter factors through the trivial subgroup.

- (Hecke period) For any normalized cusp form $f$ ($a_1 = 1$), $L(f, s) = \int_0^\infty f(it)t^s dt$. $H = \mathbb{G}_m = \text{diag}\{*, 1\} \hookrightarrow G = \text{PGL}_2$. In automorphic language, $L(\pi, s) = \int_{[H]} f(h)|h|^{s-1/2} dh$ (center $s = 1/2$).

- (Waldspurger period) $G = \text{PGL}_2$, $H' = (\text{Res}_{k'/k}\mathbb{G}_m)/\mathbb{G}_m$ a non-split torus. Then $|\int_{[H']} \phi|^2 = \frac{L(\pi,1/2)L(\pi\otimes \eta_{E/F},1/2)}{L(\pi,\text{Ad},1)}$.

- (Whittaker period) Fourier coefficients are also integrals. $X = (G/N, \psi)$. 
Examples

- (Rankin-Selberg) $L(\pi_1 \times \pi_2, s) = \int_{[GL_2]} f_1(g) f_2(g) E(g, s) dg$. 
  $G = GL_2 \times GL_2, X = \mathbb{A}^2 \times GL_2$.

- (Tate thesis) $L(\chi_p, s) = \int_{GL_1(F)} 1_{\text{Mat}_{1 \times 1}(O)}(x) \chi_p(x) |\det(x)|^s d^\times x$ for unramified $\chi_p$. $G = GL_1, X = \mathbb{A}^1$.

- (Godement-Jacquet) $G = GL_n \times GL_n, X = \text{Mat}_{n \times n}$: 
  $L(\pi_p, \text{Std}, s) = \int_{GL_{n \times n}(F)} 1_{\text{Mat}_{n \times n}(O)}(x) \langle \phi_1(x), \phi_2 \rangle |\det(x)|^s d^\times x$ for unramified $\pi_p$.

You see more examples beyond homogeneous $X = G/H$. The main player is the $G$-variety $X$ ($H$ will be the stabilizer of the open $G$-orbit).
Slogan: For any "nice" (quasi-affine spherical) $G$-variety $X$, one can construct a local $X$-period integral, and a Langlands dual group $G_X^\vee$ over $\mathbb{C}$ with a distinguished map $\iota : G_X^\vee \to G^\vee$ (see Part 2).

Conjecture: There is a local period integral $|P_X|_{2\pi}$ such that $|P_X|_{2\pi} \neq 0$ if and only if there exists a functorial lifting $\phi: \pi \to G_X^\vee$. There exists an (graded) algebraic rep $\rho_X : G_X^\vee \to \text{GL}(V_X)$ such that $|P_X|_{2\pi} = (\ast L(\sigma, \rho_X, s_0)) = (\ast L_X(\pi_v))$. 

$\Rightarrow$ images of local Langlands transfer is related to local branching laws i.e. what $G$-reps will occur in $L^2(X)$. 

$\Rightarrow$ local period integrals "= local $L$-values. Also, there shall exist precise relative character identities relating relative characters $\pi$ and $\sigma$ as distributions.
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Conjecture

- There is a local period integral $|P_X|^2_\pi$, such that $|P_X|^2_\pi \neq 0$ iff there is a functorial lifting $\phi_\sigma$ of $\pi$ to $G^\vee_X$.
- There exists an (graded) algebraic rep $\rho_X : G^\vee_X \to GL(V_X)$ such that $|P_X|^2_\pi = (*)L(\sigma, \rho_X, s_0) = (*)L_X(\pi_v)$. 
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Also, there shall exist precise relative character identities relating relative characters $\pi$ and $\sigma$ as distributions.
If $X = G/H$, $|P_X|^2$ is the natural pairing $(v_1, v_2) \mapsto \int_H \langle h.v_1, v_2 \rangle \, dh$, $v_1 \in \pi, v_2 \in \pi^\vee$. But $\rho_X$ is mysterious.

To get $|P_X|^2$ and $L(-, \rho_X, s)$ in general, the idea is to study Plancherel decomposition of $L^2(X(F))$ (or $C_c^\infty(X(F))$) under the action of $G(F)$. 
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**Conjecture**

$$L^2(X) \cong \int_{\widehat{G}_X} \iota_*(\sigma) \oplus m(\sigma) d\mu_{G_X}(\sigma),$$

where $\mu_{G_X}(\sigma)$ denotes the Plancherel measure of $G_X$ and $m(\sigma)$ is a multiplicity space.

The unramified spectrum $C_c^\infty(X(F))^G(O)$ is already interesting (related to relative Satake).

The IC function "$1_{X(O)}$" $\sim$ local unramified $L$-function, hence global (incomplete) $L$-function by products.
Part 2: spherical varieties
Spherical varieties

Now $G$ is a reductive group with a Borel $B$ over a field $k_0$, $X$ is a normal $G$-variety over $k_0$. In practice, $F$ is a local field with residue field $k_0 = \mathbb{F}_q$. For simplicity, now $k_0 = \mathbb{C}$, $F = k_0((t))$.

$X$ is spherical if $X$ has an open dense $B$-orbit $X^\circ$.

We denote by $X^\bullet$ the $G$-orbit containing $X^\circ$. $X^\bullet \cong H\backslash G$ for some $H$. 
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- Toric varieties for $G = T$ a torus. $G = \mathbb{G}_m$, $X = \mathbb{A}^1$.
- Flag variety $G/B$
- (Whittaker) $X = G/U$.
- **Fundamental example:** $X = H$, $G = H \times H$ (group case).
- More generally, symmetric spaces $X = G/K$, $K = G^\theta$.
- $G = \text{SL}_2$ on $X = \mathbb{A}^2$, $X^\bullet = \mathbb{A}^2 \setminus \{0\} = G/U$.
- (GGP) $G = SO_n \times SO_{n+1}$, $H = SO_n$. 

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Why spherical $X$? Many examples, good combinatorics, strong finiteness..
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**Geometry:** many things on $G$ can be generalized to spherical $X$: root lattice $\Delta_X$, weight lattice $\Lambda_X$, Weyl group $W_X$, dual group $G_X^\vee$, Chevalley isomorphism...
Why spherical $X$? Many examples, good combinatorics, strong finiteness..

**Geometry:** many things on $G$ can be generalized to spherical $X$: root lattice $\Delta_X$, weight lattice $\Lambda_X$, Weyl group $W_X$, dual group $G_X^\vee$, Chevalley isomorphism...

**Rep theory:** you can do geometric harmonic analysis on $X(F)$: study of $L^2(X(F))$, Fourier transform, asymptotics (or nearby cycles), Satake isomorphism...

- If $X$ is affine, $k_0[X]$ is a multiplicity-free $G$-module.
- $X$ has only finitely many $B$-orbits.
- In practice (wavefront condition),
  \[ \dim \text{Hom}_{G(F)}(\pi, C^\infty(X(F))) < +\infty: \text{uniqueness of Whittaker model for } GL_n, \dim_{\mathbb{C}} \text{Hom}_{SO_{n-1}}(\pi_{SO_n}, 1) \leq 1. \]
Classically, the action of $H \times H$ on $k[H]$ encodes $\text{Rep}(H)$ hence everything. The $B_H \times B_H$-action will encode root datum. As we don’t assume $X$ is affine, it’s better to work with fraction field $k_0(X) = k_0(X^\circ)$.
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- Weight lattice: Let $k_0(X)^{(B)}$ be the $B$-eigenfunctions in $k_0(X)$. The weight lattice $\Lambda_X \subseteq \Lambda_G$ consists of $B$-eigencharacters in $k_0(X)^{(B)}$.

- Cartan torus of $X$: $T_X = \text{Spec} \Lambda_X$. $T_X = T_G/\text{Im}(B \cap H)$ acts freely on $X^\circ$.

- The cone $\mathcal{V}$ generated by anti-dominant weights in $\Lambda_{X,Q}^\vee$: let $\mathcal{V}$ denote the set of $G$-invariant valuations $k_0(X)^\times \to \mathbb{Q}$. Evaluating a valuation on $B$-eigenfunctions gives an injective map $\mathcal{V} \to \Lambda_{X,Q}^\vee$. The advantage of considering valuations is to generalize the notion of prime divisors birationally.
Root datum: Borel action is the key

- **Weight lattice** $\Lambda_X$: $k_0(X)^{(B)} = \text{the } B\text{-eigenfunctions in } k_0(X)$. $\Lambda_X \subseteq \Lambda_G$ consists of $B$-eigencharacters in $k_0(X)^{(B)}$. Similarly, define $\Lambda_X^{++}$ for $k[X]^{(B)}$ if $X$ is quasi-affine, then $\Lambda_X = \mathbb{Z}[\Lambda_X^{++}]$. $\Lambda_X = k_0(X)^{(B)},^\times/k_0^\times$ (mult one).

- Cartan torus: $T_X = \text{Spec } \Lambda_X$.

- The polyhedral cone $\mathcal{V} \subseteq \Lambda_X^{\vee,\mathbb{Q}}$: the set of $G$-invariant valuations $k_0(X)^\times \to \mathbb{Q}$.

- Spherical roots $\Sigma_X \subseteq \Lambda_X$: generators of (extremal rays of $-\mathcal{V}^\vee) \cap \Lambda_X$.

- Normalized spherical root $\Delta_X$: integral issues, $\Sigma_X = \Delta_X$ in many cases e.g Hecke, GGP..

- Dual group $G_X^{\vee}$ over $\mathbb{C}$: given by the root datum $(\Lambda_X, \Lambda_X^{\vee}, \Delta_X, \Delta_X^{\vee})$.

Fact: up to $\{1, 2, 1/2\}$, any spherical root of $X$ is sum of two roots of $G$. 
Some generalizations

Classically, $W_G = N(T)/T$. $W_X$ is defined as the group generated by the reflections about the codimension-1-faces of the valuation cone $\mathcal{V}(X)$. It’s the Weyl group of $G^\vee_X$.

Chevalley restriction theorem:

$$\mathfrak{g}^* // G \cong \mathfrak{a}^*_G // W_G.$$ 

Example: $\text{Mat}_{n\times n} // \text{GL}_n = \mathbb{A}^n // S_n$.

Spherical variety version:

$$\mathfrak{g}_X^* // G \cong \mathfrak{a}_X^* // W_X.$$ 

Cartan decomposition

$G(O)$-orbits in $X^\bullet(F)$ is in bijection to $\Lambda^\vee_X / W_X$. 

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The dual group does not determine $X$, because it only depends on $X^\bullet$.

**Fact:** in many interesting cases (e.g. $X$ is strongly tempered),

$G_X^\vee = G^\vee$.

To classify $X$, we need colors of $X$. In a dual way, you can think a

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To classify $X$, we need colors of $X$. In a dual way, you can think a $B$-eigenfunction $f$ via the $B$-stable divisor $\text{div}(f)$.

- $\mathcal{D}(X)$ is the finite set of all $B$-stable prime divisors of $X$.
- **A color of $X$** is a $B$-stable but not $G$-stable prime divisor of $X$, and $\mathcal{D} = \mathcal{D}(H\backslash G)$ is the set of colors.
• $\rho_X : \mathcal{D}(X) \to \Lambda_X^\vee : D \mapsto v_D$: any $D \in \mathcal{D}(X)$ gives a valuation on $k(X)^\times$ hence on $\Lambda_X$. $\rho_X$ is similar to $\mathcal{V} \to \Lambda_X^\vee, \mathbb{Q}$, but may not be injective.

• The rational cone $\mathcal{C}_0 = \mathcal{C}_0(X) \subseteq \Lambda_X^\vee, \mathbb{Q}$ generated by $\rho_X(\mathcal{D}(X))$. $\text{Hom}(\Lambda_X^{++}, \mathbb{Z}_{\geq 0}) = \mathcal{C}_0 \cap \Lambda_X^\vee$. 
In group case, $G^\vee_X = H^\vee$, and colors of $H$ are in bijection to simple roots of $H$ by the Bruhat decomposition.
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In Hecke case $G = \text{PGL}_2$, $H = \mathbb{G}_m = \text{diag}\{*, 1\}$, $X = H\backslash G$, we have $G_X^\vee = G^\vee = \text{SL}_2$.

To see this, $B$-orbits on $X$ is the same as $H = \mathbb{G}_m$-orbits on $G/B = \mathbb{P}^1$, so three orbits $\mathbb{G}_m, 0, \infty$, and two colors $D^+(0), D^-(\infty)$. 
Examples

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Note $k_0[\mathbb{G}_m \backslash \text{SL}_2]^{(B)} = k_0[\mathbb{G}_m \backslash (\mathbb{A}^2 - 0)]^{(\mathbb{G}_m)}$.

$k_0[\mathbb{G}_m \backslash (\mathbb{A}^2 - 0)] = k_0[xy]$, you see the $T$-eigenvalues for $\mathbb{G}_m \backslash \text{SL}_2$ are generated by $t \mapsto t^2$, the simple root of $\text{SL}_2$.

For $X = \mathbb{G}_m \backslash \text{PGL}_2$, things are dual, so $\Lambda_X^\vee = \Lambda_G^\vee = \mathbb{Z} \frac{1}{2} \alpha^\vee$, where $\alpha : t \mapsto t$ is the simple root of $\text{PGL}_2$. And $v_{D^+} = v_{D^-} = \frac{1}{2} \alpha^\vee$. 
The rank of spherical roots is easy to compute, and is called the rank of $X$.

One can classify all rank 1 cases. Beyond the group case, there are more examples.

For example, $X = SO_{2n-1} \backslash SO_{2n}$ has $G^\vee_X = \text{PGL}_2$. $SO_2 \backslash SO_3$ is the Hecke case as before.

$X \cong \{ x \in V_{2n} | (x, x) = 1 \}$ is a "sphere" (maybe a motivation for the name "spherical varieties").

$C_c^\infty(V_{2n}) \to C_c^\infty(X(F))$ is surjective, so one can use Weil representation and theta correspondence tools.
Homogeneous spherical varieties $H\backslash G$ are classified by combinatorial invariants called homogeneous spherical datum. 

For a simple root $\alpha$ of $G$, let $B \subseteq P_\alpha$ denote the corresponding sub-minimal parabolic of $G$.

For any $B$-orbit closure $Y \subseteq X$, we say that $\alpha$ moves $Y$ if $P_\alpha Y \neq Y$.

Let $D(\alpha)$ denote the set of colors in $D$ such that $\alpha$ moves $D$.

Using this, one can describe those spherical roots of $X$ that come from $G$. 

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Theorem [Lun97, 3.2 and 3.4]

\( \#D(\alpha) \leq 2 \). There are 4 cases:

1. \( D(\alpha) = \emptyset \). Equivalently, \( \alpha \) is among the simple roots associated to the stabilizer \( P(X^\circ) \subseteq G \) of \( X^\circ \).

2. (Type U, \( SL_2/U \)) \( D(\alpha) = \{D\} \), and no multiple of \( \alpha \) is in \( X \). In this case \( v_D = \alpha^\vee|_{\Lambda_X} \).

3. (Type N, \( PGL_2/O_2 \)) \( D(\alpha) = \{D\} \), and some non-trivial multiple of \( \alpha \) is in \( X \). In this case \( v_D = \frac{1}{2}\alpha^\vee|_{\Lambda_X} \), and \( 2\alpha \in \Sigma \).

4. (Type T, \( PGL_2/G_m \)) \( D(\alpha) = \{D_\alpha^+, D_\alpha^-\} \). Equivalently, \( \alpha \in \Sigma_X \), and \( v_{D_\alpha^+} + v_{D_\alpha^-} = \alpha^\vee|_{\Lambda_X} \).
Roughly speaking, the homogeneous spherical datum associated to $X = H \backslash G$ consists of the lattice $X$; the colors $D(\alpha)$ for $\alpha \in \Sigma \cap \Delta_G$ i.e. in case 4; the set $\Sigma_X$, and the set of simple roots moving no color.
Homogeneous spherical datum

Roughly speaking, the homogeneous spherical datum associated to $X = H \backslash G$ consists of the lattice $X$; the colors $D(\alpha)$ for $\alpha \in \Sigma \cap \Delta_G$ i.e. in case 4; the set $\Sigma_X$, and the set of simple roots moving no color.

**Next step:** Then one needs to classify all spherical embeddings $X^\bullet = H \backslash G \hookrightarrow X$ for fixed $X^\bullet$.

**Origin:** classification of toric varieties by families of cones: firstly do affine toric varieties, then glue.

**Fact** (using normality): any spherical variety $X$ is covered by quasi-affine $G$-stable open subsets.
Assume $H \backslash G$ is quasi-affine. Assume $X$ is affine spherical, so $X$ has an unique closed $G$-orbit $Y$.

Let $\mathcal{C}(X) \subseteq \mathcal{C}_0(X)$ be the cone in $\Lambda^\vee X, \mathbb{Q}$ generated by the valuations $v_D$ for all $B$-stable divisors $D \in \mathcal{D}(X)$ containing $Y$. Then
Assume $H \backslash G$ is quasi-affine. Assume $X$ is affine spherical, so $X$ has an unique closed $G$-orbit $Y$.

Let $C(X) \subseteq C_0(X)$ be the cone in $\Lambda^\vee_X, \mathbb{Q}$ generated by the valuations $v_D$ for all $B$-stable divisors $D \in \mathcal{D}(X)$ containing $Y$. Then

\[ \text{[Kno91, Theorems 3.1 and 6.7]} \]

$X \mapsto C(X)$ gives a bijection (up to iso) between affine spherical embeddings of $X^\bullet$ and admissible rational polyhedral cones in $\Lambda^\vee_X, \mathbb{Q}$.

In short, the colors $\mathcal{D} = \mathcal{D}(H \backslash G)$ plus the cone $C(X)$ (admissible colored cone) give a complete understanding of all quasi-affine spherical varieties $X$. Then the full classification follows by gluing.
$n > 1$, $G = \text{GL}_n \times \text{GL}_n$, $H = \begin{pmatrix} \text{GL}_{n-1} & \ast \\ 0 & 1 \end{pmatrix}$. $H \backslash G = \text{GL}_n \times (\mathbb{A}^n - 0)$ quasi-affine but not affine.

There are $(3n - 3)$-colors and the dual group $G_X^\vee = G^\vee$. For a simple root of $\text{GL}_n$, the set $D(\alpha_i; 0) \cup D(0, \alpha_i)$ has cardinality 3 and there are no other overlaps.

Let $H \backslash G \hookrightarrow X = \text{GL}_n \times \mathbb{A}^n$ be the canonical affine embedding. The cone $\mathcal{C}(X) \cap \mathcal{V} \subset \Lambda_X^\vee, \mathcal{Q} = \mathbb{Q}^n \times \mathbb{Q}^n$ corresponds to $-\mathbb{Q}_{\geq 0}$ diagonally embedded inside $\mathbb{Q}^n \times \mathbb{Q}^n$. 
To do harmonic analysis on $X(F)$, a trick is to degenerate $X$ to simple spherical variety $X_{\emptyset}$.

A $G$-variety $X_{\emptyset}$ is horospherical if for each $x \in X_{\emptyset}$, its stabilizer subgroup in $G$ contains the unipotent radical of a Borel subgroup of $G$.

**Fact:** If $X_{\emptyset}$ is horospherical and spherical, then its dual group is always the dual torus $T_{X_{\emptyset}}$. 

There exists a principal degeneration $X \to \mathbb{A}^1$ degenerating $X$ to a horospherical variety $X_{\emptyset}$.

Idea: deformation to the normal cone.
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**[SV, 2.5]**

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Idea: deformation to the normal cone.
Part 3: Ichino-Ikeda
Globally, we have a candidate $P_H$. If $P_H \neq 0$, it’s necessary that all local spaces $\text{Hom}_{H_v}(\pi_v, 1) \neq 0$.
Consider $H = SO_n \hookrightarrow G = SO_n \times SO_{n+1}$.

**Gan-Gross-Prasad conjecture**

- (local) Whether $\text{Hom}_H(\pi, 1) \neq 0$ can be understood by $\epsilon$-factors/genericity of $\sigma$.
- (global) under local non-vanishing assumptions, globally we have $L(\pi, 1/2) \neq 0 \Leftrightarrow \exists \phi \in \pi, \int_{[H]} \phi \neq 0$. 
Examples

$G = \text{PGL}_2, H = \mathbb{G}_m$. For a cusp eigenform $f$, its central $L$-value satisfies $\left( \int_{[N]} f(n) \psi(n) dn \right) L(\pi, 1/2) = \int_{[H]} f(h) dh$. Rankin-Selberg gives $| \int_{[N]} f(n) \psi(n) dn |^2 = \prod_v \int_{N(F_v)} \langle \pi(h)f, f \rangle \psi(n) dn$. 

Hecke periods
Examples

$G = \text{PGL}_2$, $H = \mathbb{G}_m$. For a cusp eigenform $f$, its central $L$-value satisfies $(\int_{[N]} f(n)\psi(n)dn)L(\pi, 1/2) = \int_{[H]} f(h)dh$. Rankin-Selberg gives $|\int_{[N]} f(n)\psi(n)dn|^2 = \prod_v \int_{N(F_v)} \langle \pi(h)f, f \rangle \psi(n)dn$. For $v$ where everything is unramified, $\int_{N(F_v)} \langle \pi(h)f, f \rangle \psi(n)dn = \frac{1}{L(\pi_v, \text{Ad}, 1)}$. So we see

$$|\int_{[H]} f(h)dh|^2 = (\ast) \prod_v |P_v(\phi_v)|^2$$

where $|P_v(\phi_v)|^2 = \frac{L(\pi_v, 1/2)L(\overline{\pi_v}, 1/2)}{L(\pi_v, \text{Ad}, 1)}$ for unramified places.

The square absolute value of the period integral shall have a precise formula, related to central $L$-values of $\pi$. This is Ichino-Ikeda conjecture, generalizing Waldspurger’s formula or Hecke’s formula ($n = 1$).
Global Ichino-Ikeda type conjectures

For a good pair \((G, H)\), \(\dim \text{Hom}_{H_v}(\pi_v, 1) \leq 1\). The global period integral decomposes to a tensor product of local linear functionals, uniquely up to scalar.

The global period integral gives a global pairing
\[ P^{\text{Aut}} : \pi \otimes \overline{\pi} \rightarrow \mathbb{C}, \quad P^{\text{Aut}}(\phi_1, \phi_2) := \int_{[H]} \phi_1 dh \int_{[H]} \overline{\phi_2} dh = P_X(\phi_1)\overline{P_X(\phi_2)}. \]

The local Plancherel decomposition gives a local pairing
\[ P_{X,\pi_v} : \pi_v \otimes \overline{\pi_v} \rightarrow \mathbb{C}, \quad P_{v}^{\text{Planch}}(u_1, u_2) = \int_{H(k_v)} \langle \pi_v(h)u_1, u_2 \rangle du, \]

Ichino-Ikeda conjecture (imprecise)

\[ P^{\text{Aut}} = (*) \prod_v P_{v}^{\text{Planch}} \]

with a formula for (*).
Normalization is needed for the convergence of Euler products, we can normalize unramified local term to be 1.

More precisely, one computes $P_{v}^{Planch}$ for spherical unit vectors, it’s

\[ (*) \frac{L_{X}(\pi_v, 1/2)}{L(\pi_v, \text{Ad}, 1)}. \]

\[ P_{v}^{Planch, \ast} := ((*) \frac{L_{X}(\pi_v, 1/2)}{L(\pi_v, \text{Ad}, 1)})^{-1} P_{v}^{Planch}. \]
[SV] gives an conjectural generalization of local Plancherel formula, and the Ichino-Ikeda conjecture: for $\phi = \otimes_v \phi_v \in \pi = \otimes_v \pi_v$,

$$|P_{X,\pi}(\phi)|^2 = c(\pi) \cdot \frac{L_X(\pi, 1/2)}{L(\pi, \text{Ad}, 1)} \cdot \prod_v |P_{X,\pi_v}^*(\phi_v)|^2$$
[SV] gives an conjectural generalization of local Plancherel formula, and the Ichino-Ikeda conjecture: for $\phi = \otimes_v \phi_v \in \pi = \otimes_v \pi_v$,

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Examples: original Ichino-Ikeda in the GGP case, Rallis inner product formula..

- The incomplete global $L$-values $\frac{L_X(\pi,1/2)}{L(\pi, \Ad,1)}$ is defined by analytic continuation. So the local and global normalization by central $L$-values don’t cancel trivially.
- The adjoint $L$-values occur, as the normalization is based on Petersson inner products.
- $c(\pi) =$ products of some measure normalization constants and a power of 2 (size of Vogan $L$-packet).
- We ignore multiplicity $>1$ issues.
What do we know?

For $H = U_n \hookrightarrow G = U_n \times U_{n+1}$, $L_X(\pi, 1/2) = L(\pi, \text{Std}, \frac{1}{2})$, it is proved using relative trace formula after the work of many: Jacquet-Rallis, Z. Yun, W. Zhang...
For $H = SO_n \hookrightarrow G = SO_n \times SO_{n+1}$...
Main references

Thank you!