The Galois action on $KSp$ and on CM abelian varieties

Zhiyu Zhang

MIT Juivtop seminar

April 6th, 2021
Outline

1. Goal

2. Recollections
Goal

- Study principal polarized abelian varieties (PPAV) $A$ over $\mathbb{C}$ with CM by the maximal order in $E = \mathbb{Q}[\mu_q]$ ($q = p^n$ odd).

Why?
Study principal polarized abelian varieties (PPAV) $A$ over $\mathbb{C}$ with CM by the maximal order in $E = \mathbb{Q}[\mu_q]$ ($q = p^n$ odd).

Why?

$C_q \curvearrowright A \rightsquigarrow$ a map $B(\mathbb{Z}/q) \to \mathcal{A}_g(\mathbb{C}) \to \mathcal{SP}(\mathbb{Z})$ between groupoids $\rightsquigarrow$ a map $\mathbb{Z}/q[\beta] \to \KSp_*(\mathbb{Z}; \mathbb{Z}/q)$.

Bott element

$\beta \in \pi_2^s(BC_q; \mathbb{Z}/q) \cong H_2(BC_q; \mathbb{Z}/q) \hookrightarrow H_1(BC_q; \mathbb{Z}/q) = \mathbb{Z}/q$.

CM classes := image of powers of $\beta$, which generate all of $\KSp_*(\mathbb{Z}; \mathbb{Z}/q)$.
If we understand the action of $\sigma \in \text{Aut}(\mathbb{C})$ on these PPAVs $\sim$ on CM classes by **Galois equivariance**. $\sim$ on $\text{KSp}_*(\mathbb{Z}; \mathbb{Z}/q) / \text{KSp}_*(\mathbb{Z}; \mathbb{Z}_p)$.

Although $\text{KSp}_*(\mathbb{Z}; \mathbb{Z}/q)$ is small, they may contain interesting torsion information.

It’s time to do computation.
Galois equivariance

\( \text{Aut}(\mathbb{C}) \cong \text{KSp}, \text{Aut}(\mathbb{C}) \cong \mathcal{A}_g(\mathbb{C}). \)

\( \mathcal{A}_g(\mathbb{C}) \) is just a groupoid in sets, not \( \mathcal{A}^{an}_g(\mathbb{C}). \)

The map \( \mathcal{A}_g(\mathbb{C}) \to \mathcal{SP}(\mathbb{Z}) \) given by \( A \mapsto H_1(A(\mathbb{C}); \mathbb{Z}) \)

\( \rightsquigarrow \) a group homomorphism

\( \pi_{4k-2}^s (|\mathcal{A}_g(\mathbb{C})|; \mathbb{Z}/q) \to \text{KSp}_{4k-2}(\mathbb{Z}; \mathbb{Z}/q) \) which is equivariant for the action of \( \text{Aut}(\mathbb{C}). \)

For \( g = \varphi(q) = p^{n-1}(p - 1), \) it’s surjective.
Let $E$ be a CM field of degree $2g$.
If $A$ is a PPAV of dim $g$ over $\mathbb{C}$ with CM by $O_E$, then $\text{Lie} A$ is an $E \otimes \mathbb{R}$ module.
\[ \sim \text{ an } \mathbb{R}\text{-linear isomorphism } \Phi : E \otimes \mathbb{R} \cong \mathbb{C}^g. \]
A CM type of $E$ is an isomorphism $\Phi : E \otimes \mathbb{R} \cong \mathbb{C}^g$
\[ = \text{ a subset } \Phi \subseteq \text{Hom}(E, \mathbb{C}) \text{ such that } \text{Hom}(E, \mathbb{C}) = \Phi \coprod c(\Phi). \]
Main theorem of CM

Fix a CM type $\Phi : E \otimes \mathbb{R} \cong \mathbb{C}^g$. 

- $a \mapsto \mathbb{C}^g/a$ induces a bijection:
  \[ \pi_0 \left( \text{Pic}(O_E) \right) \cong \left\{ \text{PPAV } / \mathbb{C} \text{ with CM by } O_E \text{ of type } \Phi \right\} / \sim. \]

- They are all defined over $H$, the Hilbert class field of $E \rightsquigarrow$ the Galois action factors through $\text{Gal}(H/\mathbb{Q})$.

- There exists an ideal $a_\sigma$, such that $\sigma(\mathbb{C}^g/a) \cong \mathbb{C}^g/(a \otimes a_\sigma)$ for all $a$.

- A precise formula for $a_\sigma$ under Artin isomorphism:
  \[ \pi_0 \left( \text{Pic}(O_E) \right) \xrightarrow{\text{Art}} \text{Gal}(H/E). \]
Main theorem of CM

Fix a CM type $\Phi : E \otimes \mathbb{R} \cong \mathbb{C}^g$.

- $a \mapsto \mathbb{C}^g/a$ induces a bijection:
  $\pi_0 \left( \text{Pic}(O_E) \right) \cong \{ \text{PPAV} / \mathbb{C} \text{ with CM by } O_E \text{ of type } \Phi \} / \sim$.

- They are all defined over $H$, the Hilbert class field of $E \hookrightarrow$ the Galois action factors through $\text{Gal}(H/\mathbb{Q})$.

- There exists an ideal $a_\sigma$, such that $\sigma(\mathbb{C}^g/a) \cong \mathbb{C}^g / (a \otimes a_\sigma)$ for all $a$.

- A precise formula for $a_\sigma$ under Artin isomorphism:
  $\pi_0 \left( \text{Pic}(O_E) \right) \xrightarrow{\text{Art}} \text{Gal}(H/E)$.
Here we omit polarizations for PPAV.
To be precise, we need to consider an ideal \( \mathfrak{a} \) with a skew-Hermitian form \( (x, y) \mapsto \text{Tr}_E(xu \bar{y}) \) on \( \mathfrak{a} \), where \( u \) is a purely imaginary element in \( E \).
This can be explained using some groupoids.
Ring $O$ with an involution $x \rightarrow \bar{x}$.

$\omega$ any rank 1 projective $O$-module, with an $O$-linear involution $\iota : \omega \rightarrow \bar{\omega}$.

The groupoid $\mathcal{P}(O, \omega, \iota)$:

- **Objects**: pairs $(L, b)$ where $L$ is a rank 1 projective $O$-module and an isomorphism $b : L \otimes_O \bar{L} \rightarrow \omega$ s.t $b(x \otimes y) = \iota(b(y \otimes x))$.

- **Morphisms**: $O$-linear isomorphisms preserving Hermitian forms.
\( \mathcal{P}_E^+ = \mathcal{P}(O_E, \omega = O_E, c) \) is the groupoid of Hermitian forms on \( O_E \).
\( \mathcal{P}_E^- = \mathcal{P}(O_E, \omega = \delta_E^{-1}, -c) \) is the groupoid of skew-Hermitian forms on \( O_E \) valued in the inverse different.
\( \mathcal{P}_{E \otimes \mathbb{R}}^- \) is the groupoid of skew-Hermitian forms on \( E \otimes \mathbb{R} \).
Objects of \( \mathcal{P}_{E \otimes \mathbb{R}}^- \) are classified by CM types \( \Phi : E \otimes \mathbb{R} \cong \mathbb{C}^g \).
There is a map \( \pi_0(\mathcal{P}_E^-) \xrightarrow{-\otimes \mathbb{R}} \pi_0(\mathcal{P}_{E \otimes \mathbb{R}}^-) \), sending \((L, b)\) to its CM type \( \Phi_{(L, b)} \).
Shimura-Taniyama map

\( \mathfrak{a} \mapsto \Phi(O_E \otimes \mathbb{R})/\mathfrak{a} \) updates to

\[ \text{ST} : \mathcal{P}_E^- \to \mathcal{A}_g(\mathbb{C}), (L, b) \mapsto L_{\mathbb{R}}/L_{\mathbb{Z}} \]

with the **perfect symplectic pairing**

\( L_{\mathbb{Z}} \times L_{\mathbb{Z}} \mapsto \mathbb{Z}, (x, y) \mapsto -\text{Tr}_{E/\mathbb{Q}} b(x, y). \)

Φ\(_{(L,b)}\) is the unique CM type Φ such that the Hermitian form

\[ \langle x, y \rangle := -2\sqrt{-1} \sum_{j \in \Phi} j(b(x, y)) \]

is positive definite.

Perfectness by design: \( \delta_{E}^{-1} := \{ a \in E | \text{Tr}_{E/\mathbb{Q}}(ax) \in \mathbb{Z}, \forall x \in O_E \}. \)
A tensoring bifunctor $\mathcal{P}_E^- \times \mathcal{P}_E^+ \rightarrow \mathcal{P}_E^-$

We can twist PPAV with CM by $O_E$ by any ideal $a_1$ in $O_E$, $\mathbb{C}^g/a \rightarrow \mathbb{C}^g/(a_1 \otimes a)$.

To be precise, such twisting is given by a tensoring bifunctor $\mathcal{P}_E^- \times \mathcal{P}_E^+ \rightarrow \mathcal{P}_E^-$:

$$(L', b') := (L, b) \otimes (X, q) = (L \otimes_{O_E} X, b \otimes q).$$
Serre’s tensor construction

$\mathcal{P}_E^{+\text{pos.def.}} \subset \mathcal{P}_E^+ = \text{the full subgroupoid on the positive definite } (X, q) \text{ i.e i.e } q(x, x) > 0, \forall x \neq 0 \in X.$

The action by positive definite $(X, q)$ will not change CM types, and can be described via Serre’s tensor construction:

$$\text{ST } (L', b') \cong X \otimes_O \text{ST}(L, b).$$

Recall for any projective $O$-module $X$, and AV $A$ with $O$-action, $X \otimes_O A$ is the functor $R \mapsto X \otimes_O A(R)$. 
Action of complex conjugation $c$

Tensoring with $(X, q) = (O_E, -1)$, where $-1$ is the form $x \otimes y \mapsto -x \overline{y}$?

It sends $A = ST(L, b)$ to its complex conjugate variety $\overline{A}$, changing the CM type $\Phi_A$ to its complement $c(\Phi_A)$.

$\overline{A} \cong A$ as AVs, not as PPAVs with $O_E$-action.

Explanation: $A_g(\mathbb{C}) \cong \mathbb{H}_g/Sp_{2g}(\mathbb{Z})$, but complex conjugation $c$ doesn’t preserve $\mathbb{H}_g$.

How about general $\sigma \in \text{Aut}(\mathbb{C})$?
For simplicity, assume $E/\mathbb{Q}$ Galois e.g $E = K_q = \mathbb{Q}[\zeta_p]$. Then $H/\mathbb{Q}$ is also Galois.

Choose for each $\tau \in \text{Hom}(E, \mathbb{C})$ an extension $w_\tau : H \to \mathbb{C}$ to a complex embedding of $H$, such that

$$w_{\tau \sigma} = w_{c\tau} = cw_\tau.$$

Then for each $\sigma \in \text{Gal}(H/\mathbb{Q})$ and $\tau \in \text{Hom}(E, \mathbb{C})$, $w_{\sigma \tau}^{-1} \sigma w_\tau \in \text{Gal}(H/E)$.

Note if $\sigma \in \text{Gal}(H/E)$, then $w_{\sigma \tau} = w_\tau$. $(w_{\sigma \tau}^{-1} \sigma w_\tau|_H)$ is just $\sigma$ under an different embedding.
Main theorem today

\[ \sigma \mapsto F_\sigma : \mathcal{P}_E^- \to \mathcal{P}_E^- \]

**Theorem**

- On each fiber of \( \pi_0(\mathcal{P}_E^-) \to \pi_0(\mathcal{P}_E^- \otimes \mathbb{R}) \) (i.e. fixing \( \Phi = \Phi_{(L,b)} \)), \( \pi_0(F_\sigma) \) is given by tensoring certain \([(X, q)] \in \pi_0(\mathcal{P}_E^+)\), determined by \( \sigma \) and the CM type \( \Phi \).
- \( \text{Art}([X]) = \left[ \sum_{\tau \in \Phi} w_{\sigma \tau}^{-1} \sigma w_\tau \right] \) in \( \text{Gal}(H/E) \).
- If \( E = K_q \), then \( \pi_* (F_\sigma) : \pi_*^s (|\mathcal{P}_E^-|, \mathbb{Z}/q) \to \pi_*^s (|\mathcal{P}_E^-|, \mathbb{Z}/q) \) is \( \mathbb{Z}/q[\beta] \)-linear.

Slogan: Galois action is always given by twisting certain ideals, and they match under Artin isomorphism (up to summation under different embeddings).
Proof by ST formula

Part (1) is a reformulation of the ST formula in [Mil07, Theorem 4.2].
Serre tensor construction commutes with Galois action, so for each positive definite \((X, q) \in \mathcal{P}_E^+\) we have

\[ F_\sigma((L, b) \otimes (X, q)) \simeq F_\sigma(L, b) \otimes (X, q), \]

naturally in \((L, b)\) and \((X, q)\). So \(\pi_* (F_\sigma)\) is linear over the graded ring \(\pi_* (|\mathcal{P}_E^{+\text{pos.def.}}|; \mathbb{Z}/q)\).
Linearity

Serre tensor construction commutes with Galois action, so for each positive definite \((X, q) \in \mathcal{P}_E^+\) we have

\[
F_\sigma((L, b) \otimes (X, q)) \cong F_\sigma(L, b) \otimes (X, q),
\]
naturally in \((L, b)\) and \((X, q)\).

So \(\pi_*(F_\sigma)\) is linear over the graded ring \(\pi^s_*(|\mathcal{P}_E^{+\text{pos.def.}}|; \mathbb{Z}/q)\).

The map \(\mathbb{Z}/q[\beta] \to \pi^s_*(|\mathcal{P}_E^-|, \mathbb{Z}/q)\) factors through

\[
\mathbb{Z}/q[\beta] \to \pi^s_*(|BC_q|; \mathbb{Z}/q) \to \pi^s_*(|\mathcal{P}_E^{+\text{pos.def.}}|; \mathbb{Z}/q) \to \pi^s_*(|\mathcal{P}_E^-|, \mathbb{Z}/q)
\]
as \(\mathbb{Z}/q = U_1(O_q)\) is the automorphism group of \((X_0, q_0) = (O_q, 1)\).

Hence it’s also \(\mathbb{Z}/q[\beta]\)-linear.
Thank you!