Galois category and Riemann existence theorem

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Exodromy seminar

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Outline

1. Galois category
2. Reconstruction
3. Riemann existence theorem
Goal

- Define the Galois category of a scheme $X$ (via stratified shape theory).
- $\text{Gal}(X)$ can recover the étale homotopy type of $X$.
- (Riemann existence theorem) The analytic and algebraic version can be compared.
Galois 1-category of a scheme

$X$ a coherent i.e qcqs scheme $\leadsto \text{Gal}(X)$:

- **Object $x$**: geometric points $x \to X$.
- **Morphism $x \to y$**: étale specialization $y \leadsto x$ i.e a lift of $y$ to the strict localization $X(x) = \text{Spec}(O_{X,x_0}^{sh}) \to X$. 
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$X^{Zar}$ is a poset: $x_0 \leq y_0$ if and only if $x_0 \in \overline{\{y_0\}}$.

$\rightsquigarrow$ a functor $\text{Gal}(X) \to X^{Zar}: x \mapsto x_0$, fiber $BG_{\kappa(x_0)}$ over $x_0$.

$\text{Gal}(X)$ globalizes absolute Galois groups of points of $X$. 
Profinite topology on $\text{Gal}(X)$

$\text{Gal}(X)$ has a topology, like the profinite topology on $G_\kappa(x_0)$.

Idea: use finite level points $u \to X$.

An open basis of $\text{Gal}(X)$: $y \rightsquigarrow x$ lying over a given specialization $v \rightsquigarrow u$.

Can be precise using pyknotic/condensed math.
Theorem

Topological category $\text{Gal}(X)$ can recover the étale homotopy type of $X$ (up to pro-truncation), hence $\pi^\text{et}_*(X, x)$.

Idea: Stratified profinite shape can recover the profinite shape by inverting all morphisms.
∞-category: ...

Topos: the category of sheaves on a site.

∞-topos: an ∞-category $X$ satisfying ∞-Giraud’s axiom.

Geometric morphism: a pair of adjoints $(f^*, f_*) : X \to Y$ s.t $f^*$ is exact.

$S$: the ∞-category of spaces (animas).

$\text{Top}_\infty$: the ∞-category of ∞-topos.

the ∞-category $\text{Pt}(X) := \text{Fun}^*(S, X_{et})$ of points of $X$: geometric morphisms $S \to X$.

For us, let $X_{et}$ be the ∞-topos of étale sheaves valued in $S$ on the 1-site $X^e_{et}$ of étale $X$-schemes. $X_{et}$ is 1-localic.
In $\infty$-topos theory, the category of finite sets is replaced by the $\infty$-category of $\pi$-finite spaces $S_\pi$.
A lisse object $F \in X = a$ locally constant sheaf of $\pi$-finite spaces that can be trivialized on a finite cover $Y \to X$.
$X^{\text{lisse}} \subseteq X$: full subcategory of lisse objects, which is a bounded $\infty$-pretopos.
Constructible $= \text{lisse over a stratification of } X.$
Given an $\infty$-topos $X \in \text{Top}_\infty$, Lurie constructed a pro-$\infty$-groupoid $\Pi_\infty(X) \in \text{Pro}(\mathcal{S})$ called the shape of $X$. If $X$ is from a nice topological space, $\Pi_\infty(X)$ is the $\infty$-fundamental groupoid of $X$. 
Stone duality: profinite sets = totally disconnected compact Hausdorff topological spaces.

∞-Stone duality: \( S^\wedge_\pi := \text{Pro}(S_\pi) \to \text{Top}_\infty \) is fully faithful, with a left adjoint \( \hat{\Pi}_\infty : \text{Top}_\infty \to \text{Pro}(S_\pi) \) (profinite shape). Essential images are called Stone \( \infty \)-topoi.
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Construction of \( \widehat{\Pi}_\infty \): a "profinite" completion.
For a \( \pi \)-finite space \( X \), \( X \simeq \widehat{\Pi}_\infty(X) \) e.g \( \mathbb{RP}^\infty \simeq B(\mathbb{Z}/2) \).
By design, any quasi-equivalence \( X \to Y \) is a shape-equivalence.
Étale homotopy type of a scheme

$X$ is a locally noetherian scheme. Artin–Mazur defined the étale homotopy type of $X \in \text{Pro}(h_1S)$. Friedlander refined it to étale topological type of $X \in \text{Pro}(S)$. 

$\widehat{\Pi}^\text{et}_\infty (X) :=$ the profinite étale topological type.

(Hoyois) $\widehat{\Pi}^\text{et}_\infty (X) \simeq \widehat{\Pi}_\infty (X_{\text{et}})$.

- $\widehat{\Pi}^\text{et}_\infty (\text{Spec}(k)) = BG_k$.
- $\widehat{\Pi}^\text{et}_\infty (\mathbb{C}\mathbb{P}^1) = (S^2)^{\wedge}_\pi$. 
Bounded coherent $\infty$-topoi can be classified via $\infty$-pretopoi. 
[SAG, Theorem E.2.3.2] For any $\infty$-topos $X$, $\text{Sh}_{\text{eff}}(X^{\text{lisse}}) \in \mathbf{Top}^{\text{Stone}}_{\infty}$ (effective epimorphism topology) is called Stone reflection of $X$, $\text{Sh}_{\text{eff}}(X^{\text{lisse}}) \leftrightarrow \hat{\Pi}_{\infty}(X)$.

$\infty$-Stone duality $\sim \text{Fun}\left(\hat{\Pi}_{\infty}(X), S_\pi\right) \simeq X^{\text{lisse}}$.

In particular for qcqs noetherian scheme $X$, 

$$\text{Fun}\left(\hat{\Pi}_{\infty}^{\text{et}}(X), S_\pi\right) \simeq X^{\text{lisse}}_{\text{et}}.$$ 

**Next step**: define a stratified version of $\hat{\Pi}_{\infty}(X_{\text{et}})$. 

$P$ a finite poset.
A $P$-stratified space $X = \text{an } \infty$-category $X$ with a conservative functor $X \to P$.

Hochster duality: profinite posets $=$ spectral topological spaces.
$
\rightsquigarrow
S$-stratified spaces for any spectral topological space $S$.

$\text{Str}_\pi = \text{the } \infty$-category of $\pi$-finite stratified spaces.
$S$-stratified $\infty$-topos $=$ an $\infty$-topos $X$ equipped with a geometric morphism $X \to \text{Sh}(S)$ to the $\infty$-topos of sheaves of spaces on $S$. 
Theorem

\[ \text{Pro}(\text{Str}_\pi)_S \leftrightarrow \text{StrTop}_{\infty,S} \] extending

\[ [\Pi \to P] \mapsto [\text{Fun}(\Pi, S) \to \text{Fun}(P, S)] \]

is fully faithful, with a left adjoint

\[ \hat{\Pi}^S_{(\infty,1)} : \text{StrTop}_{\infty,S} \to \text{Pro}(\text{Str}_\pi)_S \] (profinite S-stratified shape).

Essential images are called spectral \( \infty \)-topoi.
Similar to Stone reflection, there is a spectralification functor

\[ \text{StrTop}_{\infty,S} \to \text{StrTop}^{\text{spec}}_{\infty,S} \overset{\hat{\Pi}^S_{(\infty,1)}}{\cong} \text{Pro}(\text{Str}_\pi)_S, \quad X \mapsto \text{Sh}_{\text{eff}}(X^{S\text{-cons}}). \]
For any \(S\)-stratified \(\infty\)-topos \(X\), adjunction gives a natural equivalence:

\[
\text{Fun}\left(\hat{\Pi}^{S}_{(\infty,1)}(X), S_{\pi}\right) \cong X^{S-\text{cons}}.
\]

The \(\infty\)-category of representations of \(\hat{\Pi}^{S}_{(\infty,1)}(X)\) valued in \(\pi\)-finite spaces = \(S\)-constructible sheaves on \(X\).
Return to the coherent scheme $X$, $S := X^{Zar}$, $\rightsquigarrow$ stratified $\infty$-topos $X^{et} \to X^{Zar}$. It’s a spectral $\infty$-topos. Profinite stratified étale homotopy type $\hat{\Pi}^{et}_{(\infty,1)}(X) := \hat{\Pi}^{X^{Zar}}_{(\infty,1)}(X_{et})$.

**Theorem**

\[ \text{Gal}(X) \simeq \hat{\Pi}^{X^{Zar}}_{(\infty,1)}(X_{et}). \]

**Corollary**

\[ \text{Fun} \left( \text{Gal}(X), S_\pi \right) \simeq X^{\text{cons}}_{\text{ét}}. \]
Reconstruction

**Idea:** A constructible sheaf $\mathcal{F}$ is lisse iff all specializations of $\mathcal{F}$ are isomorphisms.

**Homotopy theorem**
For any spectral $S$-stratified $\infty$-topos $X$, the profinite classifying space of $\hat{\Pi}_S^{(\infty,1)}(X)$ is precisely $\hat{\Pi}_\infty(X)$.

In particular, there is an equivalence $\theta_X : \hat{\Pi}_\infty^\text{ét}(X) \to \varepsilon(\text{Gal}(X))$. This finishes reconstruction theorem, let’s see some examples.
An example

We use the language of spatial décollages.

\[ X = \mathbb{A}^1_{\mathbb{C}}, \quad P = [0 \to \infty, 1 \to \infty], \] a stratification \( X \to P \) given by

\[ X(0) = Z_0 = \{0\}, \quad X(1) = Z_1 = \{1\}, \quad X(\infty) = U = \mathbb{A}^1_{\mathbb{C}} - \{0, 1\}. \]

\[ \text{Gal}^P(X) \to P. \]

- \( \text{Gal}^P(X)(0) = \hat{\Pi}_\infty(X(0)) = B\{\ast\}. \)
- \( \text{Gal}^P(X)(1) = \hat{\Pi}_\infty(X(1)) = B\{\ast\}. \)
- \( \text{Gal}^P(X)(\infty) = \hat{\Pi}_\infty(X(\infty)) = BF(x_0, x_1) \) the classifying
groupoid for profinite completion of the free group of two
variables.
- \( \text{Gal}^P(X)(0 \to \infty) = \hat{\Pi}_\infty(X(x_0) \setminus \{x_0\}) = B\hat{\mathbb{Z}}. \)
- \( \text{Gal}^P(X)(0) \leftarrow \text{Gal}^P(X)(0 \to \infty) \to \text{Gal}^P(X)(\infty). \)
Another example

Let \((A, K, k)\) be a DVR, \(S = \text{Spec}A, s = \text{Spec}k, \eta = \text{Spec}K\). 
\(S_{et}\) is a naturally [1]-stratified spectral \(\infty\)-topos, with closed stratum \(s_{et}\) and open stratum \(\eta_{et}\).

\[
s_{et} \times_{S_{et}} S_{et} = S_{et}^h. \quad s_{et} \times_{S_{et}} \eta_{et} = \eta_{et}^h.
\]

**Example**

\[
\hat{\Pi}_\infty^{\text{ét}}(\eta) \simeq B\Gamma_K, \quad \hat{\Pi}_\infty^{\text{ét}}(\eta^h) \simeq BD_A,
\]

\[
\hat{\Pi}_\infty^{\text{ét}}(\eta^{sh}) \simeq BI_A, \quad \hat{\Pi}_\infty^{\text{ét}}(S^h) \simeq BG_k.
\]

\(BG_k \leftarrow BD_A \rightarrow BG_K\).
Let $K$ be a number field, and write $O_K$ be the ring of integers of $K$.

$\text{Gal}(O_K)$ has objects (up to iso) the prime ideals of $O_K$.

The profinite stratified etale shape of $\text{Spec} O_K$ is stratified by the various closed strata, each of which is an embedded profinite "circle" $BG_{k(p)} \cong \hat{\mathbb{Z}}$ i.e a knot.

Enveloping each knot is a tubular neighborhood, given by $\text{Gal}(\text{Spec} O_{p}^{sh})$. And the deleted tubular neighborhood is given by $BG_{Kp}$. 
Riemann existence theorem

\( X \) a finite type \( \mathbb{C} \)-scheme. 
\( X^{an} = \) complex points of \( X \) with analytic topology. 
SGA4 \( \leadsto \) a geometric morphism of 1-localic \( \infty \)-topoi

\[ \varepsilon_{X,*} : X^{an} \to X_{et} \]

s.t for any \( f : X \to Y \), we have \( f^{et}_{*} \varepsilon_{X,*} \simeq \varepsilon_{Y,*} f^{an}_{*} \).
Riemann existence theorem

Riemann Existence Theorem

\[ \varepsilon_{X,*} \text{ restricts to an equivalence} \quad X^\text{lisse}_{\text{ét}} \simeq X^\text{lisse}_{\text{an}} \quad \text{between} \]
\[ \infty\text{-categories of lisse sheaves}. \]

Equivalently, it induces an equivalence of profinite spaces

\[ (X^{\text{an}})^{\wedge}_\pi = \hat{\Pi}_\infty (X^{\text{an}}) \simeq \hat{\Pi}_\infty (X^{\text{et}}). \]
Note $\varepsilon_{X,*} : X_{an} \to X_{et}$ is over $S = X^{Zar}$ i.e $S$-stratified, the pullback functor $\varepsilon^{X,*}$ restricts to a morphism of $\infty$-pretopoi:

$$\varepsilon^{X,*} : X^{S-\text{cons}}_{et} \to X^{S-\text{cons}}_{an}.$$ 

$$(X/S)_{an} := \text{Sh}_{\text{eff}}\left(X^{S-\text{cons}}_{an}\right), \quad (X/S)_{et} := \text{Sh}_{\text{eff}}\left(X^{S-\text{cons}}_{et}\right).$$ 

$$\Rightarrow \varepsilon_{X,*} : (X/S)_{an} \to (X/S)_{et}.$$ 

**Proposition 12.6.4 in [Exo]**

The pullback functor $\varepsilon^{X,*}$ restricts to an equivalence on constructible sheaves.
Proof by reduction

Idea: reduce to lisse version by gluing. Do induction for dimension of $X$. If $\dim = 0$, then constructible=lisse, done. Write $X^{Zar}$ as limits of $S = Z^{Zar} \cup \{\infty\}$.

$$(Z/Z^{Zar})_{an} \xrightarrow{i_*} (X/S)_{an} \xleftarrow{j^*} (U/\infty)_{an}.$$ 

$Z_{et} \xrightarrow{i_*} (X/S)_{et} \xleftarrow{j^*} (U/\infty)_{et}.$

An $\infty$-topos $X$ can be recovered from a closed subtopos $Z$, its open complement $U$, and the gluing information in the deleted tubular neighborhood $W$ of $Z$ in $U$. $W = Z \times_X U$ (oriented fiber product).
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$$(X/S)_{et} \leftrightarrow (U/\infty)_{et}.$$  

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$\epsilon$ is natural, i.e $f^\an \epsilon^X, * F \simeq \epsilon^Y, * f^\et F$ holds for any constructible sheaf $F \in X_{et}$.

$\leadsto$ the gluing data are also matched, we’re done.
Van Kampen Theorem

If $X = Z \cup^\phi U$ is a bounded coherent constructible $[1]$-stratified $\infty$-topos. Then the pushout of the morphisms

$\hat{\Pi}_\infty(Z \times_X U) \to \hat{\Pi}_\infty(Z)$, $\hat{\Pi}_\infty(Z \times_X U) \to \hat{\Pi}_\infty(U)$ is exactly $\hat{\Pi}_\infty(X)$.
The natural morphism $\epsilon : \text{Gal}_{an}(X) \to \text{Gal}(X)$ is an equivalence. 
$\text{Gal}_{an}(X)$ is related to the exit path category of $X^{an}$ in topology.
An anabelian application

Let $k$ be a finitely generated field of characteristic 0. Then a normal $k$-variety $X$ can be reconstructed from the stratified homotopy type of $(X \otimes_k \bar{k})^{an}$ with its action of $G_k$. 
Thank you!