$D^b_{m}(X_0, \overline{\mathbb{Q}}_\ell)$ and Weil II

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BBDG seminar

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Outline

1. Weights
2. Weil II
3. Applications: ABCDE
4. Descent
Goals

- Set up weight theory, and $D^b_m(X_0, \mathbb{Q}_\ell)$.
- Recall Weil II, apply it to show $D^b_m(X_0, \mathbb{Q}_\ell)$ and $Perv_m(X_0, \mathbb{Q}_\ell)$ have nice properties.
- Prove the key vanishing of higher $Ext^i$ (yoga of weights).
- State Frobenius "descent": $\text{Perv}(X_0) \leftrightarrow \text{Perv}(X, Fr_q)$. 
Notations

1. \((-)_0/\mathbb{F}_q \sim (-)/k = \mathbb{F}_q^{alg}.

2. \(\Lambda = \overline{\mathbb{Q}}_\ell \ (\ell \neq p).

3. A lisse \(\overline{\mathbb{Q}}_\ell\)-sheaf = a continuous finite dim \(\overline{\mathbb{Q}}_\ell\)-rep of \(\pi_1\).

4. \(p_{1/2} =\) selfdual perversity i.e \(p D^0_c = \{K | \dim \text{Supp} \mathcal{H}^i(K) \leq -i\} \).
What is the Frobenius $Fr_q$?

$X_0$ scheme of finite type over $\mathbb{F}_q$, $\mathcal{F}_0$ a $\overline{\mathbb{Q}}_\ell$-sheaf on $X_0$.

$\text{Gal}_{\mathbb{F}_q} \cong \hat{\mathbb{Z}}$ is generated by geometric Frobenius $F = (a \mapsto a^{1/q})$.

**Example**

On $\text{Spec} \mathbb{F}_q$, a $\overline{\mathbb{Q}}_\ell$-sheaf $\equiv$ a vector space $V$ with a linear automorphism $F$.

Easy way: $x \in X_0(\mathbb{F}_q)$, pullback of $\mathcal{F}_0$ along $x$ gives a vector space $\mathcal{F}_{\tilde{x}}$ with $\text{Gal}_{\mathbb{F}_q}$-action.

$Fr_{q,x}^* := F \sim \mathcal{F}_{\tilde{x}}$. 
What is the Frobenius $Fr_q$?

Another way:

$k$-linear **relative Frobenius** $Fr_q = Fr_{X/k} : X \rightarrow X$, with fixed points $X(\mathbb{F}_q)$. $Fr_X = (Fr_k)_X \circ Fr_q$.

**Example**

$X_0 = \mathbb{A}^1 = \text{Spec} \mathbb{F}_q[t]$, $(Fr_k)_X : a_it^i \mapsto a_i^q t^i$, $Fr_q : a_it^i \mapsto a_it^{iq}$.
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**Example**

$X_0 = \mathbb{A}^1 = \text{Spec} \mathbb{F}_q[t]$, $(Fr_k)_X : a_it^i \mapsto a_q^it^i$, $Fr_q : a_it^i \mapsto a_it^{iq}$.

- Absolute Frob doesn’t change etale topology, $Fr_X^* \mathcal{F} = \mathcal{F}$.
- $\mathcal{F}$ comes from $\mathcal{F}_0$, $(Fr_k)_X^* \mathcal{F} \cong \mathcal{F}$.

$\leadsto$ a natural isomorphism $\phi : Fr_q^* \mathcal{F} \cong \mathcal{F}$.

$\leadsto Fr_q^* = \phi_x \circ \mathcal{F}_{\bar{x}}$ for any $x \in X_0(\mathbb{F}_q)$. $Fr_q^* = Fr_q^* x$. 
For [BBDG], \( w \in \mathbb{Z} \).

- weak \textbf{q-Weil number} of weight \( w \): \( a \in \overline{\mathbb{Q}} \) s.t \( |\nu(a)| = q^{w/2} \), \( \forall \nu : \overline{\mathbb{Q}} \rightarrow \mathbb{C} \).
Punctual purity

For [BBDG], $w \in \mathbb{Z}$.

- **weak $q$-Weil number** of weight $w$: $a \in \overline{\mathbb{Q}}$ s.t $|\iota(a)| = q^{w/2}$, $\forall \iota: \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

- A $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}_0$ on $X_0$ is **punctually pure of weight** $w$, if for any $n$ and $x \in X_0(\mathbb{F}_{q^n})$, the eigenvalues of $Fr_{q^n}$ on $\mathcal{F}_x$ are all weak $q^n$-Weil numbers of weight $w$.

$\overline{\mathbb{Q}}_\ell(n)$ is puncturally pure of weight $-2n$. 
Mixedness for a sheaf

\( \mathcal{F}_0 \) is **mixed**, if \( \exists \) a finite filtration on \( \mathcal{F}_0 \) with **punctually pure** successive quotients.

\( w(\mathcal{F}_0) := \) punctural weights of \( \mathcal{F}_0 \).

By definition, mixedness is stable under extensions, subquotients. This **mixed condition** is geometric and motivic.
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**This mixed condition is geometric and motivic.**

We can also require \( a \in \overline{\mathbb{Z}} \) i.e a \( q \)-Weil number.

[Weil II, Remark 1.2.8] and [KW]: \( \iota \)-mixedness, uses \( \iota : \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C} \), \( w \in \mathbb{R} \).

[Weil II, Conj 1.2.9]: every \( \overline{\mathbb{Q}}_\ell \)-sheaf on \( X_0 \) is \( \iota \)-mixed (\( w \in \mathbb{R} \)). This is known by works of L. Lafforgue, V. Drinfeld.
Mixedness for a complex: $D^b_m(X_0, \overline{\mathbb{Q}}_\ell)$

$D^b_m(X_0, \overline{\mathbb{Q}}_\ell) \subseteq D^b_c(X_0, \overline{\mathbb{Q}}_\ell)$ consists of $K_0$ s.t the cohomology sheaves $\mathcal{H}^i K_0$ are mixed.

By long exact seq for $\mathcal{H}^i$, $D^b_m$ is stable under extensions.
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[BBDG, 5.1.6-5.1.7] (to be proved later)

- $D^b_m(X_0, \overline{\mathbb{Q}}_\ell)$ is stable under $\mathbb{D}$.
- $D^b_m(X_0, \overline{\mathbb{Q}}_\ell)$ inherits the perverse t-structure.
- Every sub-quotient of a mixed perverse sheaf is mixed.

Lisse $\mathcal{F}_0$ on smooth $X_0$: 😊😊

Gluing is non-trivial, we need Weil II (e.g $Rj_*\;$ for open immersion $j$).
$D^b_{\leq w}, D^b_{\geq w} \subseteq D^b_m$

$K_0$ in $D^b_m$ is of weight $\leq w$ if punctural weights of $\mathcal{H}^i K_0$ are $\leq w + i$ for any $i$.

$K_0$ in $D^b_m$ is of weight $\geq w$ if its Verdier’s dual $\mathbb{D}(K_0)$ of weight $\leq -w$.

$K_0$ is **pure of weight** $w$ if $K_0 \in D^b_{\leq w} \cap D^b_{\geq w}$. By duality,
\(D^b_{\leq w}, D^b_{\geq w} \subseteq D^b_m\)

\(K_0\) in \(D^b_m\) is of weight \(\leq w\) if punctual weights of \(\mathcal{H}^i K_0\) are \(\leq w + i\) for any \(i\).

\(K_0\) in \(D^b_m\) is of weight \(\geq w\) if its Verdier’s dual \(\mathbb{D}(K_0)\) of weight \(\leq -w\).

\(K_0\) is pure of weight \(w\) if \(K_0 \in D^b_{\leq w} \cap D^b_{\geq w}\). By duality,

[BBDG,5.1.9]

\(K_0 \in D^b_m\) is in \(D^b_{\leq w}\) (resp \(D^b_{\geq w}\)), iff for any closed point \(i : x_0 \to X_0, i^* K_0\) (resp \(i^! K_0\)) is of weight \(\leq w\) (resp \(\geq w\)).

\(D^b_{\leq w}[1] = D^b_{\leq w+1}\). But \(D^b_{\leq w} \cap D^b_{\geq w+1} = 0\) (to be proved later) is non-trivial. We need Weil II.
Purity—punctual purity for lisse $\mathcal{F}_0$ on smooth $X_0$

Assume $X_0$ is smooth of pure dimension $d$, so $\omega_{X_0} = \overline{\mathbb{Q}}_\ell[2d](d)$. If $\mathcal{H}^i K_0$ are all lisse, then

Proposition

$$\mathcal{H}^i(DK_0) = (\mathcal{H}^{-2d-i} K_0)^\vee(d).$$

Proof:
Assume $X_0$ is smooth of pure dimension $d$, so $\omega_{X_0} = \overline{Q}_\ell[2d](d)$. If $\mathcal{H}^i K_0$ are all lisse, then

**Proposition**

$$\mathcal{H}^i(\mathbb{D}K_0) = (\mathcal{H}^{-2d-i} K_0)^\vee(d).$$

**Proof:**

$$\mathbb{D}(\mathcal{H}^i(K_0)) = R\text{Hom}(\mathcal{H}^i(K_0), \overline{Q}_\ell[2d](d)) = R\text{Hom}(\mathcal{H}^i(K_0), \overline{Q}_\ell)[2d](d)$$

(lisse so higher local Ext sheaves $= 0$)

$$= \text{Hom}(\mathcal{H}^i(K_0), \overline{Q}_\ell)[2d](d) = \mathcal{H}^i(K_0)^\vee[2d](d).$$

$$\sim E_2^{pq} = \mathcal{H}^p(\mathbb{D}(\mathcal{H}^{-q} K_0)) \Rightarrow \mathcal{H}^{p+q}(\mathbb{D}K_0) \text{ degenerates},$$

$$\mathcal{H}^i(\mathbb{D}K_0) = \mathcal{H}^{-2d}\mathbb{D}(\mathcal{H}^{-2d-i}(K_0)).$$
Purity—punctual purity for lisse $\mathcal{F}_0$ on smooth $X_0$

**Proposition**

Assumption as above, $K_0$ is pure of weight $w$ iff each $\mathcal{H}^i K_0$ is punctually pure of weight $w + i$.

$$-(w - 2d - i) - 2d = w + i.$$ 

In general, $Q_l$ is puncturally pure but may not be pure. If $X_0$ is proper, and $Q_l$ is pure of weight 0, then Frob eigenvalues on $H^i(X)$ has weights exactly $i$ by Weil II, which is not true in general.

It can be pure in some singular cases.
If $f : X_0 \to Y_0$ is a (separated) morphism between schemes of finite type over $\mathbb{F}_q$, then $Rf_!$ sends $D_{\leq w}^b$ to $D_{\leq w}^b \subseteq D_m^b$.

Corollary [Weil II, 6.1] (induction + proper case + smooth case)

$Rf_*$ sends $D_m^b$ to $D_m^b$. 

Weil II

Example

\[ a : X_0 \to \mathbb{F}_q \xrightarrow{\sim} \text{Frob eigenvalues on } H^i_c(X, \mathcal{F}) \text{ are } q\text{-Weil numbers of weight } \leq w + i \text{ for any } \mathcal{F}_0 \in D^b_{\leq w}(X_0, \overline{\mathbb{Q}_\ell}). \]

Example

\[ X \subseteq \mathbb{P}^{d+1} \text{ is a smooth geometrically irreducible hypersurface over a finite field } \mathbb{F}_q. \text{ Then } \#X(\mathbb{F}_{q^n}) - \#\mathbb{P}^d(\mathbb{F}_{q^n}) = O(q^{nd/2}). \]

Now it’s time for applications to \( D^b_m \).
Application A: stabilities of $D^b_m$

By definition, $f^*$ preserves $D^b_{\leq w}$, and $\otimes$ sends $D^b_{\leq w} \times D^b_{\leq w'}$ to $D^b_{\leq w+w'}$.

**Proposition**

$K_0 \in D^b_m \iff \mathbb{D}(K_0) \in D^b_m$.

**Proof**: WLOG $K_0$ is a mixed sheaf. If $X_0$ is smooth, $K_0$ is lisse, then $\mathbb{D}K_0 = K_0^\vee[2d](d)$ by previous computation.
Application A: stabilities of $D^b_m$

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$$\mathbb{D}K_0 = K^\vee_0[2d](d)$$

by previous computation.

In general, use Noetherian induction. Choose smooth dense open

$j : U_0 \hookrightarrow X_0$ s.t $j^*K_0$ is lisse.

Exact triangle $j_!j^*K_0 \to K_0 \to i_*i^*K_0$.

$$\sim i_*\mathbb{D}(i^*K_0) \to \mathbb{D}(K_0) \to j_*\mathbb{D}(j^*K_0).$$

We’re done by induction and Weil II for $Rj_*$. 
Application A: stabilities of $D^b_m$

By duality and Weil II, we get

- $f^!$ sends $D^b_{\leq w}$ to $D^b_{\leq w}$.
- $f_*$ sends $D^b_{\geq w}$ to $D^b_{\geq w}$.
- $\otimes$ sends $D^b_{\leq w} \times D^b_{\leq w}'$ to $D^b_{\leq w} + w'$. 
- $R\text{Hom}$ sends $D^b_{\leq w} \times D^b_{\geq w}'$ to $D^b_{\geq -w} + w'$.

$R\text{Hom}(A, D(B)) = R\text{Hom}(A, R\text{Hom}(B, \omega X_0)) = R\text{Hom}(A \otimes B, \omega X_0) = D(A \otimes B)$. 
By duality and Weil II, we get

**[BBDG,5.1.14]**

- \(D \) exchanges \( D_{\leq w}^b \) and \( D_{\geq -w}^b \).
- \( f_! , f^* \) sends \( D_{\leq w}^b \) to \( D_{\leq w}^b \).
- \( f^! , f_* \) sends \( D_{\geq w}^b \) to \( D_{\geq w}^b \).
- \( \otimes \) sends \( D_{\leq w}^b \times D_{\leq w'}^b \) to \( D_{\leq w+w'}^b \).
- \( R\text{Hom} \) sends \( D_{\leq w}^b \times D_{\geq w'}^b \) to \( D_{\geq -w+w'}^b \).

\[
R\text{Hom}(A, D(B)) = R\text{Hom}(A, R\text{Hom}(B, \omega_{X_0})) = R\text{Hom}(A \otimes B, \omega_{X_0}) = D(A \otimes B).
\]
Therefore, the full subcategory $D^b_{m}(X_0, \overline{\mathbb{Q}}_\ell)$ of $D^b_c(X_0, \overline{\mathbb{Q}}_\ell)$ is stable by all usual operations e.g $Rf_*, Rf^!, f^*, Rf^!$, $\otimes$, $R\text{Hom}$, $\mathbb{D}$. 
Application B: perverse $t$-structure on $D^b_m$

[BBDG, 5.1.7. (i)]

$D^b_m$ is stable under $p\tau_{\leq i}$ and $p\tau_{\geq i}$.

$D^b_m$ is stable under $\tau_{\leq i}$ and $\tau_{\geq i}$ by definition.

$p\tau_{\leq 0}(-)$ is constructed by $j_*, j^*, i_*, i^*$ plus truncations and taking cones.

More precisely, $F_1 \to F \to Rj_* p\tau_{\geq 1} j^* F$, $p\tau_{\leq 0}(F) \to F_1 \to Ri_* p\tau_{\geq 1} i^* F_1$.

So $p\tau_{\leq i}$, $p\tau_{\geq i}$ send $D^b_m$ to $D^b_m$, we’re done.

$\sim D^b_m$ is stable under $pH^*$.

$\sim D^b_m$ is stable under $pj_*, pj_!, pj^*$. 
Application C: vanishing of higher $\text{Ext}^i$

$K_0, L_0 \in D^b_m(X_0, \overline{\mathbb{Q}_\ell})$.

[BBDG, 5.1.15]

- If $K_0 \in D^b_{\leq w}, L_0 \in D^b_{\geq w}$, then $\text{Hom}^i(K, L)^F = 0$, for $i > 0$.
- If $K_0 \in D^b_{\leq w}, L_0 \in D^b_{> w}$, then $\text{Hom}^i(K_0, L_0) = 0$, for $i > 0$. 
Application C: vanishing of higher $\text{Ext}^i$

$K_0, L_0 \in D_m^b(X_0, \overline{\mathbb{Q}_\ell})$.

[**BBDG, 5.1.15**]

- If $K_0 \in D_{\leq w}^b, L_0 \in D_{\leq w}^b$, then $\text{Hom}^i(K, L)^F = 0$, for $i > 0$.
- If $K_0 \in D_{\leq w}^b, L_0 \in D_{> w}^b$, then $\text{Hom}^i(K_0, L_0) = 0$, for $i > 0$.

Crucial for semi-simplicness and decomposition theorem in [BBDG].

Lack of weight theory $\sim$ decomposition theorem fails for coefficients like $\mathbb{F}_\ell$.

**Proof:** $a : X_0 \to \text{Spec} \mathbb{F}_q$, apply Weil II to

$M_0 := Ra_* R\text{Hom}(K_0, L_0) \in D_c^b(\text{Spec} \mathbb{F}_q)$. 

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Application D: $D^b_{\leq w} \cap D^b_{\geq w+1} = 0$

$$id_K \in \text{Hom}(K, K)^F = \text{Hom}^1(K, K[-1])^F = 0.$$
$M_0 \in D^b_c(\text{Spec } \mathbb{F}_q)$

$M_0 \in D^b_c(\text{Spec } \mathbb{F}_q, \overline{\mathbb{Q}}_\ell)$, $M := i^* M_0 \in D^b_c(\overline{\mathbb{Q}}_\ell)$ with geom Frobenius action $F : M \to M$.

Global section / fixed points $R\Gamma : D^b_c(\text{Spec } \mathbb{F}_q, \overline{\mathbb{Q}}_\ell) \to D^b_c(\overline{\mathbb{Q}}_\ell)$.

[BBDG, 5.1.2]

Short exact sequence $0 \to (H^{n-1} M)_F \to H^n R\Gamma M_0 \to (H^n M)_F \to 0$.

$E_2^{pq} = H^p(\text{Spec } \mathbb{F}_q, \mathcal{H}^q M_0) = H^p(\text{Gal}_{\mathbb{F}_q}, H^q M) \Rightarrow H^{p+q} R\Gamma M_0$. 
$M_0 \in D^b_c(\text{Spec } \mathbb{F}_q)$

$M_0 \in D^b_c(\text{Spec } \mathbb{F}_q, \overline{\mathbb{Q}}_\ell)$, $M := i^* M_0 \in D^b_c(\overline{\mathbb{Q}}_\ell)$ with geom Frob action $F : M \to M$.

Global section / fixed points $R\Gamma : D^b_c(\text{Spec } \mathbb{F}_q, \overline{\mathbb{Q}}_\ell) \to D^b_c(\overline{\mathbb{Q}}_\ell)$.

[BBDG, 5.1.2]

Short exact sequence $0 \to (H^{n-1}M)_F \to H^n R\Gamma M_0 \to (H^n M)_F \to 0$.

$E^{pq}_2 = H^p(\text{Spec } \mathbb{F}_q, \mathcal{H}^q M_0) = H^p(\text{Gal}_{\mathbb{F}_q}, H^q M) \Rightarrow H^{p+q} R\Gamma M_0$.

$\text{Gal}_{\mathbb{F}_q} \cong \hat{\mathbb{Z}}$, generated by the geometric Frobenius $F$.

$$H^p(\text{Gal}_{\mathbb{F}_q}, -) = \begin{cases} (-)^F, & p = 0 \\ (-)_F, & p = 1 \\ 0, & \text{else} \end{cases}$$
\[ M_0 := Ra_\ast R\text{Hom}(K_0, L_0) \in \mathcal{D}_c^b(\text{Spec } \mathbb{F}_q) \]

\[ a : X_0 \to \text{Spec } \mathbb{F}_q. \quad M_0 = Ra_\ast R\text{Hom}(K_0, L_0), \quad M = i^\ast M_0. \]

**Proposition**

\[ M = R\text{Hom}(K, L) \text{ (smooth base change).} \]

\[ R\Gamma M_0 = R\text{Hom}(K_0, L_0) \quad (R\Gamma Ra_\ast = R\Gamma X_0). \]
$M_0 := Ra_* R\text{Hom}(K_0, L_0) \in D_c^b(\text{Spec } \mathbb{F}_q)$

$a : X_0 \to \text{Spec } \mathbb{F}_q$. $M_0 = Ra_* R\text{Hom}(K_0, L_0)$, $M = i^* M_0$.

**Proposition**

$M = R\text{Hom}(K, L)$ (smooth base change).

$R\Gamma M_0 = R\text{Hom}(K_0, L_0)$ ($R\Gamma Ra_* = R\Gamma X_0$).

From above,

$0 \to (\text{Hom}^{i-1}(K, L))_F \to \text{Hom}^i(K_0, L_0) \to \text{Hom}^i(K, L)^F \to 0$.

- $K_0 \in D_{\leq w}^b, L_0 \in D_{\geq w}^b \leadsto M_0 \in D_{\geq 0}^b \leadsto w(\text{Hom}^i(K, L)) \geq i$,
  $\text{Hom}^i(K, L)^F = 0$, for $i > 0$.

- If $K_0 \in D_{\leq w}^b, L_0 \in D_{\geq w}^b$, then $M_0 \in D_{> 0}^b$. $\text{Hom}^i(K_0, L_0) = 0$, for $i > 0$.

**Application C** is proved.
Application E: $\text{Perv}_m(X_0)$ is stable under subquotient

$\text{Perv}_m(X_0) = \text{Perv}(X_0) \cap D^b_m$.

[[BBDG, 5.1.7. (ii)]]

$A_0 \subseteq B_0 \in \text{Perv}(X_0)$, $B_0 \in \text{Perv}_m(X_0) \Rightarrow A_0, B_0/A_0 \in \text{Perv}_m(X_0)$.

**Proof**: If $A_0, B_0$ are concentrated in one degree, this is obvious.
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Proof: If $A_0, B_0$ are concentrated in one degree, this is obvious.

In general, choose smooth dense open $j : U_0 \hookrightarrow X_0$ s.t $j^* A_0 \rightarrow j^* B_0$ are lisse (up to shift). Then $j^* A_0 = p j^* A_0$ is mixed and perverse.

$\sim p j^! j^* A_0, p j_* j^* (B_0/A_0) \in \text{Perv}_m(X_0)$. 

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**Proof:** If $A_0, B_0$ are concentrated in one degree, this is obvious. In general, choose smooth dense open $j: U_0 \hookrightarrow X_0$ s.t $j^* A_0 \hookrightarrow j^* B_0$ are lisse (up to shift). Then $j^* A_0 = pj^* A_0$ is mixed and perverse.

$\sim pj_!j^* A_0, pj_* j^* (B_0/A_0) \in \text{Perv}_m(X_0)$.

$\sim I = \text{Im}(pj_!j^* A_0 \to B_0), J = \text{Ker}(B_0 \to pj_* j^* (B_0/A_0)) \in \text{Perv}_m(X_0)$.

$I \subseteq A_0 \subseteq J \subseteq B_0$. $J/I \in \text{Perv}_m(Z_0)$ gives $A/I \in \text{Perv}_m(Z_0)$ by induction. $A_0$ is extension of $A_0/I$ by $I$, hence also mixed.
Let $\text{Perv}(X, Fr_q)$ be the category of perverse sheaves $\mathcal{F}$ on $X$ equipped with an isomorphism $\phi : Fr_q^* \mathcal{F} \to \mathcal{F}$. Now $\Lambda$ is any suitable coefficient.
Let $\text{Perv}(X, Fr_q)$ be the category of perverse sheaves $\mathcal{F}$ on $X$ equipped with an isomorphism $\phi : Fr_q^* \mathcal{F} \to \mathcal{F}$. Now $\Lambda$ is any suitable coefficient.

[BBDG, 5.1.2]

1. The functor $\text{Perv}(X_0) \to \text{Perv}(X, Fr_q), \mathcal{F}_0 \mapsto (\mathcal{F}, Fr_q)$ is fully faithful.

2. The category of essential images is stable by extensions and by sub-quotients.

As perversity gives a $t$-structure, $\text{Hom}^{-1}(K_0, L_0) = 0$. So $\text{Hom}_{D^b_c(X_0)}(K_0, L_0) = \text{Hom}_{D^b_c(X)}(K, L)^F$, part (1) follows.
Thank you!
Stable under extensions

\((\ast)\) \(0 \to (\text{Hom}(K, L))_F \to Ext^1(K_0, L_0) \to Ext^1(K, L)_F \to 0\), is exact, where \(Ext^1(A, B) = \{0 \to B \to C \to A \to 0\}/\simeq\).

Recall \(0 \to B \to C_1 \to A \to 0 \simeq 0 \to B \to C_2 \to A \to 0\) iff there is a \(f : C_1 \cong C_2\) such that \(f|_A = id_A, f|_B = id_B\).

The Frob action \(F\) on \(Ext^1(K, L)\) is via pullback along \(Fr_q^*\).
Stable under extensions

\((\ast)\) $0 \rightarrow (\text{Hom}(K, L))_F \rightarrow \text{Ext}^1(K_0, L_0) \rightarrow \text{Ext}^1(K, L)^F \rightarrow 0$, is exact, where $\text{Ext}^1(A, B) = \{0 \rightarrow B \rightarrow C \rightarrow A \rightarrow 0\}/\sim$.

Recall $0 \rightarrow B \rightarrow C_1 \rightarrow A \rightarrow 0 \simeq 0 \rightarrow B \rightarrow C_2 \rightarrow A \rightarrow 0$ iff there is a $f : C_1 \cong C_2$ such that $f|_A = id_A, f|_B = id_B$.

The Frob action $F$ on $\text{Ext}^1(K, L)$ is via pullback along $Fr_q^*$.

[BBDG, 5.1.2]

There is another short exact sequence \((\ast\ast)\)

$0 \rightarrow (\text{Hom}(K, L))_F \rightarrow \text{EXT}^1(((K, \phi_K), (L, \phi_L)) \overset{\text{forget}}{\rightarrow} \text{Ext}^1(K, L)^F \rightarrow 0$,

where $\text{EXT}^1 =$ extensions in $\text{Perv}(X, Fr_q)$. 

Stable under extensions

Proof:

kernel of forget is given by $(L \oplus K, \begin{pmatrix} \phi_L & U \phi \\ 0 & \phi_K \end{pmatrix})$, where the class is determined by $U$ modulo $\phi Fr_q^*(V)\phi^{-1} - V$ for $V : K \to L$. Hence the kernel is the coinvariant.

So $Ext^1(K_0, L_0) \cong EXT^1((K, \phi), (L, \phi))$ by $(\ast)$ and $(\ast\ast)$. 
Stable under subquotients

Do induction as in application E.
A singular example
References

- Notes on a learning Seminar on Deligne’s Weil II Theorem, Umich Summer 2016.
- P. Deligne, Weil II.