12/21 Next lec: Jan 8

Course ends: Feb 12

G/E $\quad \mathrm{Bun}_G : \mathrm{Perf}_{\overline{\mathbb{F}}_q} \longrightarrow$ groupoids

Thm 1) Bun_G is an Artin v-stack

2) $|\mathrm{Bun}_G| \longrightarrow B(G)$ continuous, bijection

$\forall\, b \in B(G)$, get locally closed stratum

$$\mathrm{Bun}_G^b \subseteq \mathrm{Bun}_G \qquad \mathrm{Bun}_G^b = [* / \mathcal{G}_b]$$

$$1 \longrightarrow \text{"unipotent gp diamond"} \longrightarrow \mathcal{G}_b \longrightarrow \underline{G_b(E)} \to 1$$

$\exists$

(so cohom. smooth)

iterated ext of pos BC spaces

$\rightsquigarrow \quad [* / \underline{G_b(E)}] \longrightarrow [*/\mathcal{G}_b] = \mathrm{Bun}_G^b$

cohom. smooth

$\simeq$

$\mathrm{Bun}_{G_b}^1$ cohom. smooth Artin v-stack

$\Rightarrow$ Bun_G^b also a cohom. smooth Artin v-stack

$\dim = -\langle 2\rho, \nu_b \rangle$

2ρ = sum of positive roots

<u>**Cor**</u> $\kappa : \pi_0 \mathrm{Bun}_G \cong \pi_1(G)_\Gamma$

Equivalently, each connected component of Bun_G is the closure of Bun_G^b for a unique $b \in B(G)$

proof: <u>enough</u>: Any $\neq \phi$ open substack $\mathcal{U}$ $\subseteq \mathrm{Bun}_G$ contains a basic pt

(Then $\forall$ $b \in B(G)_{\text{basic}}$, any

$\phi \neq \mathcal{U} \subseteq \kappa^{-1}(\kappa(b)) \subseteq \mathrm{Bun}_G$

open and closed

have $\mathrm{Bun}_G^b \subseteq \mathcal{U}$ $\Rightarrow$ $\kappa^{-1}(\kappa(b))$ connected

b unique basic pt

Take minimal $b \in B(G)$ st $\mathrm{Bun}_G^b \subseteq \mathcal{U}$

b minimal $\Rightarrow$ $\mathrm{Bun}_G^b \subseteq \mathcal{U}$ open

$\mathcal{U}$ coh smooth of dim 0 (it's open in Bun_G)

$\cup$| open

Bun_G^b coh smooth of dim $-\langle 2\rho, \nu_b \rangle$

$\Rightarrow$ $0 = -\langle 2\rho, \nu_b \rangle \Rightarrow \nu_b$ central

$\Leftrightarrow$ b basic $\square$

$D_{\text{ét}}(\mathrm{Bun}_G, \Lambda) \circlearrowleft$ geometric Hecke operator

$\rightsquigarrow$

L-parameters for Schur-irred. objects in $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$

rep theory of all $G_b(E)$

<u>Proposition</u> $\forall\ b \in B(G)$

$$D_{ét}(\mathrm{Bun}_G^b, \Lambda) \cong D(G_b(E), \Lambda)$$

derived category of ab cat

of smooth reps of $G_b(E)$

on Λ-mod

Pf: <u>Step 0</u> $D_{ét}(*, \Lambda) \cong D(\Lambda)$

geom pt char p

(easy: $D_{ét}(\mathrm{Spa}\, C, \Lambda) \cong D(\Lambda)$

as $\mathrm{Spa}\, C$ spatial diamond of finite

coh. dim

so $D_{ét}(\mathrm{Spa}\, C, \Lambda) \cong D(\underbrace{(\mathrm{Spa}\, C)_{ét}}_{\text{site of finite sets}}, \Lambda)$

topos = punctured topos)

But $* = \mathrm{Spa}\, \overline{\mathbb{F}_q}$ <u>not</u> a diamond need to analyze via descent along

$$\mathrm{Spa}\, C \longrightarrow \mathrm{Spa}\, \overline{\mathbb{F}}_q$$

⇝ <u>Prop</u> $\forall$ any small v-stack $/ \overline{\mathbb{F}}_q$,

C complete alg closed

pull back $D_{\acute{e}t}(X, \Lambda) \longrightarrow D_{\acute{e}t}(X \times_{\overline{\mathbb{F}}_q} \mathrm{Spa}\, C, \Lambda)$

fully faithful

$$\Rightarrow \begin{tikzcd} D_{\acute{e}t}(*, \Lambda) \arrow[rr, hook] & & D_{\acute{e}t}(\mathrm{Spa}\, C, \Lambda) \cong D(\Lambda) \\ & D(\Lambda) \arrow[ul] \arrow[ur, "\cong"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

$\Rightarrow D_{\acute{e}t}(*, \Lambda)$

$\cong D(\Lambda)$

<u>Step 1</u> $D_{\acute{e}t}([* / \underline{G_b(E)}], \Lambda)$

$\cong D(G_b(E), \Lambda)$

holds for any locally pro-p-group H in place of $\underline{G_b(E)}$

also $D_{ét}([\mathrm{Spa}\, C \,/\, \underline{G_b(E)}], \Lambda) \cong D(G_b(E), \Lambda)$

(for Step 1: <u>idea</u> use descent along

$$\mathrm{Spa}\, C \longrightarrow [\mathrm{Spa}\, C \,/\, \underline{G_b(E)}]$$

better: $D_{ét}([\mathrm{Spa}\, C \,/\, \underline{G_b(E)}], \Lambda)$

$$\cong D([\mathrm{Spa}\, C \,/\, \underline{G_b(E)}]_{ét}, \Lambda)$$

= site of sets with continuous $G_b(E)$-action

$$\cong D(G_b(E), \Lambda) : \text{ smooth } G_b(E)\text{-rep on } \Lambda\text{-mods} \,)$$

<u>Step 2</u> $D_{ét}(\mathrm{Bun}_G^b, \Lambda) \cong D_{ét}([* \,/\, \underline{G_b(E)}], \Lambda)$:

$$[*/G_b(E)] \longrightarrow \mathrm{Bun}_G^b$$

coh. smooth

fibers have trivial cohomology

$$\rightsquigarrow \quad D_{ét}(\mathrm{Bun}_G^b, \Lambda) \longrightarrow D_{ét}([*/G_b(E)], \Lambda)$$

fully faithful

$$D_{ét}([*/G_b(E)], \Lambda) \Rightarrow \text{Step 2}$$

Step 1 + Step 2 $\Rightarrow$ ✓ $\square$

Cor (of proof) $\forall\ C/\overline{\mathbb{F}}_q$ complete alg closed

$$D_{ét}(\mathrm{Bun}_G^b, \Lambda) \cong D_{ét}(\mathrm{Bun}_G^b \times_{\overline{\mathbb{F}}_q} \mathrm{Spa}\, C, \Lambda)$$

Cor $D_{ét}(\mathrm{Bun}_G, \Lambda) \cong D_{ét}(\mathrm{Bun}_G \times \mathrm{Spa}\, C, \Lambda)$

and admits an infinite semi orthogonal decomp

with pieces $D_{ét}(\mathrm{Bun}_G^b, \Lambda) \simeq D(G_b(E), \Lambda)$

<u>proof</u> Bun_G has stratification with pieces

$$i^b : \mathrm{Bun}_G^b \longrightarrow \mathrm{Bun}_G$$

functor $i^b_!$, i^{b*} etc induce semi-ortho

decomp on Bun_G and $\mathrm{Bun}_G \times_{\overline{\mathbb{F}_q}} \mathrm{Spa}\, C$:

$$\left[\begin{array}{c} Z \overset{i}{\subseteq} X \overset{j}{\supseteq} U \\ \\ D_{ét}(Z) \underset{i_*}{\overset{i^*}{\rightleftarrows}} D_{ét}(X) \underset{j^*}{\overset{j_!}{\leftrightarrows}} D_{ét}(U) \end{array}\right]$$

know : $D_{ét}(\mathrm{Bun}_G, \Lambda) \hookrightarrow D_{ét}(\mathrm{Bun}_G \times_{\overline{\mathbb{F}_q}} \mathrm{Spa}\, C, \Lambda)$

but ess image contains all $i^b_! D_{ét}(\mathrm{Bun}_G^b \times_{\overline{\mathbb{F}_q}} \mathrm{Spa}\, C, \Lambda)$

so everything $\square$

How strata intersect is encoded in those spaces

$\pi_b : \mathcal{M}_b \longrightarrow \mathrm{Bun}_G$ for last lecture

Thm 1) $D_{ét}(\mathrm{Bun}_G, \Lambda)$ is compactly generated.

and a complex

$A \in D_{ét}(\mathrm{Bun}_G, \Lambda)$ is compact

$\Longleftrightarrow$ all $(i^b)^* A \in D_{ét}(\mathrm{Bun}_G^b, \Lambda)$

are compact i.e lie in thick

triang. subcategory generated by

$c\text{-}\mathrm{Ind}_K^{G_b(E)} \Lambda$ $\quad K \subseteq G_b(E)$ open pro-p subgp

and almost all $(i^b)^* A$ are zero

Compact objects in $D_{ét}(\mathrm{Bun}_G, \Lambda)$

are not Verdier self-dual

smooth dual of $c\text{-}\mathrm{Ind}_K^{G_b(E)} \Lambda$ is uncountably dim already in pure rep theory

Problem: $c\text{-}\mathrm{Ind}_K^{G_b(E)}\Lambda$ is uncountably dim

$\dim\left(c\text{-}\mathrm{Ind}_K^{G_b(E)}\Lambda\right)^{K'} = \infty$ in general

2) On $D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda)^{\omega} \subseteq D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda)$

subcat of compact objs, have

Bernstein – Zelevinsky duality functor

$$\mathbb{D}_{BZ} : \left(D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda)^{\omega}\right)^{op} \longrightarrow D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda)^{\omega}$$

s.t. $R\mathrm{Hom}(A, B) \cong \pi_{\natural}\left(\mathbb{D}_{BZ}(A) \overset{\mathbb{L}}{\underset{\Lambda}{\otimes}} B\right)$

$\pi : \mathrm{Bun}_G \longrightarrow *$ proj

$\pi_{\natural}$ = left adjoint of π^*

(= twist of $R\pi_!$, could also look $R\pi_!$)

$\mathbb{D}_{BZ}^2 \simeq \mathrm{id}$

For $b \in B(G)$ basic, restrict to self-duality

on $D_{ét}(\mathrm{Bun}_G^b, \Lambda)^\omega \cong D(G_b(E), \Lambda)^\omega$

and it restrict to usual BZ duality

$$\mathbb{D}_{BZ}(c\text{-}\mathrm{Ind}_K^{G_b(E)} \Lambda) \cong c\text{-}\mathrm{Ind}_K^{G_b(E)} \Lambda$$

[Recall: If $\mathcal{C}$ triang cat, then $X \in \mathcal{C}$ compact if $\mathrm{Hom}_{\mathcal{C}}(X, -)$ commutes with all direct sums

Second duality functor: Gaitsgory, Drinfeld ...

3) "$D_{ét}(\mathrm{Bun}_G)$-analogue of admissibility"

$A \in D_{ét}(\mathrm{Bun}_G, \Lambda)$ is ULA (universally locally acyclic) (later) (for the map $\mathrm{Bun}_G \to *$)

iff $\forall b \in B(G)$ $(i^b)^* A$ is admissible in the classical sense i.e $\forall$ open pro-p subgp $K \subseteq G_b(E)$

$$[(i^b)^* A]^K \in D(\Lambda)$$

is perfect (rep. by finite complex of finite prj Λ-mod)

4) The class of 3) is stable under Verdier duality

$$\mathbb{D}_{\mathrm{Bun}_G}(A) = R\mathrm{Hom}(A, R\pi^! \Lambda)$$

and satisfy biduality (restrict to smooth duality on rep)

$$A \simeq \mathbb{D}_{\mathrm{Bun}_G}(\mathbb{D}_{\mathrm{Bun}_G}(A))$$

<u>Remark</u> Ideally, would like to have notion of "constructible complexes" on Bun_G; these should be the compact objs,

and universally locally acyclic for $\mathrm{Bun}_G \to *$

But this doesn't work!

Theorem is best replacement, but note

compact $\not\Rightarrow$ ULA   ULA $\not\Rightarrow$ compact

e.g. $c\text{-}\mathrm{Ind}_K^{G_b(E)} \Lambda$ compact, but not admissible ($=$ ULA)

$\bigoplus_{i=1}^{\infty} \pi_i$ (π_i supercusp irr rep of $G_b(E)$ with growing conductor)

$\Rightarrow \left(\bigoplus_{i=1}^{\infty} \pi_i\right)^K = \bigoplus_{i=1}^{N(K)} \pi_i^K$ $N(K) < +\infty$

admissible, but not compact

(Just like the space of auto forms)

Warning: there is also a notion of constructible complexes on (locally) spatial diamonds by descent on small v-stacks, (generated by $j_! \Lambda$, $j: U \to X$) qcqs étale map

But this is yet different, and almost no $A \in D_{ét}(\mathrm{Bun}_G, \Lambda)$ is constructible in that sense! (they will be loc const)

<u>Example</u>. $X = \mathbb{D}_C$ closed unit disc

$i: \mathrm{Spa}\, C \hookrightarrow X$
closed im of origin

Then $i_* \Lambda$ is <u>not</u> constructible

<u>Problem</u> $0 \to j_! \Lambda \to \Lambda \to i_* \Lambda \to 0$

but j not quasi-compact i.e

$\mathbb{D}_C^*$ not quasicompact

In fact, constructible sheaves on rigid analytic var X are locally constant in an open neighborhood of any classical pt.

Upshot: Notions of
" finitely generated "
" admissible" reps
Bernstein - Zelevinsky duality
smooth duality

generalize to
$D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda)$

Remarks on coefficients: only allowed Λ st. $n\Lambda = 0$ $(n,p)=1$

Ideally, want $\Lambda = \overline{\mathbb{Q}_\ell}$

But passage from $\mathbb{Z}/l^n\mathbb{Z}$ - coeff to

$\mathbb{Z}_l$ - coeff is more tricky than usual

can define (∞ - cat)

$$D_{et}(Bun_G, \mathbb{Z}_l) := \varprojlim_n D_{ét}(Bun_G, \mathbb{Z}/l^n\mathbb{Z})$$

this is related to reps on l-adic

complete $\mathbb{Z}_l$ - mods

$$D_{ét}(*, \mathbb{Z}_l) = \varprojlim D(\mathbb{Z}/l^n\mathbb{Z})$$

But want reps on discrete $\mathbb{Z}_l$ - r.s !

Usual trick : Use

$$\mathrm{Ind}\left(\varprojlim D_{ét}(Bun_G, \mathbb{Z}/l^n)^{\omega}\right)$$

Then $\quad \mathrm{Ind}\left(\varprojlim_n D(\mathbb{Z}/\ell^n\mathbb{Z})^{\omega}\right)$

$= D(\mathbb{Z}_\ell)$ derived ∞-cat of $\mathbb{Z}_\ell$-mod

(finite free $\mathbb{Z}_\ell$-mod are ℓ-adic complete)

But this does not work here, as compact obj are not admissible

(compact obj is not finite enough!)

Using idea of solid modules

$\rightsquigarrow$ a version of $\mathrm{Det}(\mathrm{Bun}_G, \Lambda)$

for <u>any</u> $\mathbb{Z}_\ell$-alg Λ

for which all assertions in this lecture generalizes still true.

$$\begin{tikzcd} \mathbb{P}^1 \setminus \mathbb{P}(E) = \Omega^2 \arrow[r] \arrow[d] & \mathrm{Bun}_G^{\mathcal{O}(\frac{1}{2})} \arrow[d, "\text{open}"] \\ \mathbb{P}^1 \arrow[r, "\mathrm{BL}"', "\text{unif}"] & \mathrm{Bun}_G \end{tikzcd}$$

↑ not qc

not qc

Q: BZ duality on non-basic strata

e.g. $\mathcal{O} \oplus \mathcal{O}(1)$

don't know ~ how to compute

Q: ULA ~ coh, smooth

Q: p-adic gp BZ duality

DL complex ↗ relate $(V \to V^\vee)$ dual

Maybe

Q: L-parameter $\rightsquigarrow$ Heck operation

$$T_V : D_{ét}(Bun_G, \Lambda) \longrightarrow \underline{D_{ét}(Bun_G \times Div^1, \Lambda)}$$

$\simeq D_{ét}(Bun_G, \Lambda)^{W_E}$

$$T_V = R\overrightarrow{h}_!(\overleftarrow{h}^* A \otimes q^* V)$$

$q : Heck_G \longrightarrow locHeck_G$

$\simeq L^+G \backslash Gr_G$