Bun_G 12/11

G / E red gp

Bun_G v-stack on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$

$S \longmapsto \{G\text{-bundle on } X_S\}$

Thm $|\mathrm{Bun}_G| \longrightarrow B(G)$

bijective continuous

exa GL_n ✓

One piece remaining in the proof

Let $\mathrm{Bun}_G^1 \subseteq \mathrm{Bun}_G$ substack of all G-bundle $\mathcal{E}/X_S$ s.t $\forall$ $\mathrm{Spa}(C,C^+) \longrightarrow S$ $\mathcal{E}|_{X_{\mathrm{Spa}(C,C^+)}}$ is trivial

(C: complete alg closed field)

<u>prop</u> $\mathrm{Bun}_G^1 \hookrightarrow \mathrm{Bun}_G$ is an open substack

and $\mathrm{Bun}_G^1 \cong [*/\underline{G(E)}]$

Pf: know ν semi-continuous

$$\mathrm{Bun}_G^1 \subseteq \{\mathcal{E} \mid \nu = 0\}$$

can reduce to G-bundle $\mathcal{E}/X_S$ s.t $\nu = 0$

In that case, for all reps in $\mathrm{Rep}_E G$

$$\rho : G \longrightarrow GL(V)$$

$$\rho_* \mathcal{E} \in VB(X_S) \text{ everywhere semi-stable of slope } 0$$

So $\iff$ an $\underline{E}$-local system on S

By pro-étale localization, can assume S strictly totally disconnected, let

$$A = \mathrm{Cont}(|S|, E) = \mathrm{Cont}(\pi_0 S, \mathcal{E})$$

finite proj

$$\{ \underline{E}\text{-local systems on } S \} \cong \mathrm{Proj}(A)$$

A-mod

$$\mathbb{L} \longmapsto \mathbb{L}(S)$$

So $\mathcal{E}$ defines an exact $\otimes$-functor

$$\mathrm{Rep}_E G \longrightarrow \mathrm{Proj}(A)$$

i.e a G-torsor $\mathcal{F}$ on $\mathrm{Spec}\, A$

enough to show: If $\mathcal{F}$ is trivial at

$\operatorname{Spec} E \hookrightarrow \operatorname{Spec} A$, then it's trivial

$s \in S$
valuation

after pullback to $\operatorname{Spec} \operatorname{Cont}(\mathcal{U}, E) \subseteq \operatorname{Spec} \operatorname{Cont}(|S|, E)$

for some open + closed $U \subseteq |S|$
st

This follows from two facts

1) The local ring

$\varinjlim_{U \ni s} \operatorname{Cont}(\mathcal{U}, E)$ is henselian along

$\operatorname{Ker}($evaluation at $s)$

$= \operatorname{Ker}(\quad \twoheadrightarrow E$

(for example, as local rings of analytic adic spaces like $\operatorname{Spa} A$ are always Henselian

2) If $(B, I) / E$ henselian pair

then $H^1_{et}(\operatorname{Spec} B, G) \hookrightarrow H^1_{et}(\operatorname{Spec} B/I, G)$

is injective

$\square$

Digression on local Shimura Varieties

(cf. Rapoport-Viehmann)

p-adic analogue of Shimura varieties
("local")

related to Shimura var by uniformation

Čerediek '70s Rapoport-Zink '80s

[RZ96] PEL type

in this case, relevant local Shimura varieties are moduli spaces of p-div gps with PEL type

"Rapoport-Zink spaces"

But there should be general local Shimura var

(not directly related to p-div)

Berkeley Lectures : construction of local Shimura var in general

Loc Shimura data (we can assume $E = \mathbb{Q}_p$)
by Weil restriction

triple $(G, [b], \{\mu\})$

- G/E reductive gp
- $\{\mu : \mathbb{G}_m \longrightarrow G_E\}$ conj class of minuscule cocharacter

  Q: Griffith transv

- $[b] \in B(G)$

for local Sh to be non-empty
we need $[b] \in B(G,\mu) \subseteq B(G)$

finite subset
defined combinatorically

Loc Sh Variety

Q: descent data
(+ Weil descent)

tower $\left(\mathcal{M}_{(G,b,\mu),K}\right)$ $K \subseteq G(E)$ cpt open
of smooth rigid-analytic var $/ \breve{E}$

with compatible étale period maps

$\pi_K : \mathcal{M}_{(G,b,\mu),K} \longrightarrow \mathcal{F}\ell_{\mu} \quad / \check{E}$

all non-empty geometric fibers $\cong G(E)/K$

partial flag var G/P_{μ}

defined by dynamic method

<u>Exa</u> (Drinfeld) $G = D^{\times}$ $\quad D/E$ quat alg <u>divison</u>

$\mu : \mathbb{G}_m \longrightarrow G_{\bar{E}} \cong GL_2$

$t \longmapsto \begin{pmatrix} t & \\ & 1 \end{pmatrix}$

b basic, slope $1/2$

(unique <u>basic</u> element of $B(G,\mu)$)

$$\rightsquigarrow \quad \mathcal{M}_K \longrightarrow \Omega^2 = \mathbb{P}^1 - \mathbb{P}^1(E)$$
$$\subseteq \mathcal{F}l_\mu = \mathbb{P}^1$$

they are Drinfeld covers of Ω^2

similar exa for $D^\times_{1/n}$, $n \geq 2$

Exa (Lubin-Tate)

$$G = GL_n$$

$$\mu : \mathbb{G}_m \longrightarrow G \qquad t \longmapsto \begin{pmatrix} t & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$(\mathcal{M}_K)_{K \subseteq GL_n(E)} \longrightarrow \mathcal{F}l_\mu = (\mathbb{P}^{n-1})^{ad}_{\breve{E}}$$

parametrize def of 1-dim ht $= n$ p-divs

special to this case

surjective étale map

(Gross-Hopkins period map)

fiber $= GL_n(E)/K$

So $(\mathbb{P}^{n-1})^{ad}$ admits nontrivial infinite degree étale covering spaces.

$K = GL_n(\mathcal{O}_E) \subseteq GL_n(E)$

$$\Rightarrow \quad \mathcal{M}_K \cong \coprod_{\mathbb{Z}} (n-1)\text{-dim open unit disc}$$

$$\cong \coprod_{\mathbb{Z}} \left(\operatorname{Spa} W_E(\overline{\mathbb{F}}_q)[\![u_1, \cdots, u_{n-1}]\!]\right)_{\breve{E}}$$

Construction of local Shimura Varieties

Want : open subset $\mathcal{F}\ell_\mu^{adm} \subseteq \mathcal{F}\ell_\mu$

"admissible locus"

$+$ $G(E)$ - local system $\mathbb{L}$ on $\mathcal{F}\ell_\mu^{adm}$

Then $\mathbb{L} \longrightarrow \mathcal{F}\ell_\mu^{adm}$ $G(E)$ - torsor

$$\rightsquigarrow \mathcal{M}_K = \mathbb{L}/K \longrightarrow \mathcal{F}\ell_\mu^{adm}$$

automatically étale

So as ${}^{\circ}\mathcal{F}\ell^{adm}_{\mu}$ smooth rigid analytic

$\rightsquigarrow$ $\mathcal{M}_K$ smooth rigid analytic varieties

<u>Recall</u>

$${}^{\circ}\mathcal{F}\ell^{\lozenge}_{\mu} \cong \mathrm{Gr}_{G,\le\mu} \subseteq \mathrm{Gr}_G := \mathrm{Gr}_G^{B^+_{dR}} \longrightarrow \mathrm{Bun}_G$$

μ minuscule

(If μ $\overset{\text{not}}{\vee}$ minuscule, use the diamond $\mathrm{Gr}_{G,\le\mu}$ $\overset{\text{not } =\mu\text{, better!}}{\rightsquigarrow}$ moduli of local Shtukas)

<u>Prop</u> ${}^{\circ}\mathcal{F}\ell^{\lozenge}_{\mu} \longrightarrow \mathrm{Bun}_G$

image meets Bun_G^{b} $\iff$ $b \in B(G,\mu)$

Pf: Appendix of Rapoport to "p-adic coh of Lubin-Tate tower" equiv, $\mathcal{E}_b$ can be a modification of triv G-torsor of type μ^{-1}

i.e in image of analogus map

$$
\begin{tikzcd}
{}^{\sigma}\mathcal{F}\ell_{\mu^{-1}} \arrow[r, squiggly] & \mathrm{Bun}_G
\end{tikzcd}
$$

modify $\mathcal{E}_1$ $\square$

$$
\begin{tikzcd}
{}^{\sigma}\mathcal{F}\ell_{\mu}^{\diamond} \arrow[r] & \mathrm{Bun}_G \\
({}^{\sigma}\mathcal{F}\ell_{\mu}^{adm})^{\diamond} \arrow[u, phantom, "\cup\ \text{open}"] \arrow[r] & \mathrm{Bun}_G^1 \arrow[u, phantom, "\cup\ \text{open}"] \cong [* / \underline{G(E)}]
\end{tikzcd}
$$

get the desired ${}^{\sigma}\mathcal{F}\ell_{\mu}^{adm}$

and $\underline{G(E)}$-torsor on ${}^{\sigma}\mathcal{F}\ell_{\mu}^{adm}$

Cor $\varprojlim_K \mathcal{M}_K^\diamond$ parametrizes modification

$\mathcal{E}_b \cong \mathcal{E}_1$ of type μ

more precisely $\forall\ S \in \mathrm{Perf}/\mathrm{Spa}\,\breve{E}$

$$\varprojlim_K \mathcal{M}_K^\diamond(S) = \{\text{isom } \mathcal{E}_b|_{X_S \setminus S^\#} \cong \mathcal{E}_1|_{X_S\setminus S^\#}\}$$

modification "type μ at all geo pts"

Exa Lubin-Tate case

$$\varprojlim_{K \subseteq GL_n(E)} \mathcal{M}^\diamond_{LT,K} \cong \{\mathcal{O}^n_{X_S} \hookrightarrow \mathcal{O}_{X_S}(\tfrac{1}{n})\}$$

cokernel supp at $S^\#$

(a line bundle on $S^\#$)

$$\varprojlim_{K' \subset D^\times_{\frac{1}{n}}} \mathcal{M}^\diamond_{Dr,K'} \cong$$

Isom of Lubin-Tate and Drinfeld tower

easy

now works for all local Shimura variety with b basic

$$(G, b, \mu) \rightsquigarrow (G_b, b^{-1}, \mu^{-1})$$

General pts of Bun_G

1) Semi-stable pt

Thm $\mathrm{Bun}_G^{ss} \subseteq \mathrm{Bun}_G$ open

and $\mathrm{Bun}_G^{ss} = \coprod_{b \in B(G)_{basic}} [* \,|\, \underbrace{G_b(E)}_{\mathrm{Bun}_G^b}]$

pf: open : semi-cont of ν

decomposition : K is locally const
into Bun_G^b
$$K : B(G)_{\text{basic}} \xrightarrow{\cong} \pi_1(G)_\Gamma$$

remains to show : $\mathrm{Bun}_G^b \cong [* / \underline{G_b(E)}]$

But $\mathcal{E}_b$ G-torsor on X_S

and $\underline{\mathrm{Aut}}_{X_S}(\mathcal{E}_b) \simeq G_b \times_E X_S$ for basic b

$\leadsto$ $\{G\text{-torsor on } X_S\} \cong \{G_b\text{-torsor on } X_S\}$
$$\mathcal{E} \longmapsto \underline{\mathrm{Isom}}(\mathcal{E}, \mathcal{E}_b)$$
$$\underline{\mathrm{Aut}}(\mathcal{E}_b) = G_b\text{-torsor}$$

$\leadsto$ basic b induces isom

$$\begin{array}{ccccc} \mathrm{Bun}_G & \cong & \mathrm{Bun}_{G_b} & & \\ \cup\!\!| & \circlearrowleft & \cup\!\!| & & \\ \mathrm{Bun}_G^b & \cong & \mathrm{Bun}_{G_b}^1 & \cong & [* / \underline{G_b(E)}] \end{array}$$ $\square$

2) Non-semistable non-basic b

Thm $Bun_G^b \subseteq Bun_G$ locally closed

$Bun_G^b \cong [* / \mathcal{G}_b]$

$\mathcal{G}_b$ is a gp v-sheaf

$$1 \to \underbrace{\mathcal{G}_b^0}_{} \to \mathcal{G}_b \to \underline{G_b(E)} \to 1$$

ext of pos dim BC space

$\mathcal{G}_b$ has dim $= \langle 2\rho, \nu(b) \rangle$

Can we replace G by a reductive gp over FF curve ?

(X_S for each S)

Better "reductive gp in Isoc_E" (Anschütz)

Q: Griffith transversality $\Rightarrow$ minuscule μ

p-adic ?

$$\begin{tikzcd} & \mathrm{Gr}_{G,\mu} \arrow[d, "\pi_\mu"] \\ S^\diamond \arrow[ur, dashed, "???"] \arrow[r] & {}^{0}\mathrm{Fl}_\mu^\diamond \end{tikzcd}$$

$/ (\mathrm{Spa}\, \breve{E})^\diamond$

(μ general)

π_μ Brylinski – Birula

rigid analytic var smooth

fibers are iterated aff spaces

$0 \to (\mathbb{A}^1)^\diamond \to X \to (\mathbb{A}^1)^\diamond \to 0$

X not rigid analytic var !

Thm (S, p-adic HT for rigid-analytic var, Kedlaya, Fontaine-Faltings) $\exists$

$$\mathrm{Gr}_{G,\mu}(S^\diamond) \hookrightarrow {}^{0}\mathrm{Fl}_\mu^\diamond(S^\diamond)$$

image = those maps s.t. Griffith transversality

Q : It's unknown that $\varprojlim M_K$ is perfectoid for general G.

Q : Moduli of Isoc is ∞-dim

(deformation spaces

Q : algebraization result for int models of RZ spaces

(unknown, only way to attack it) is to apply global Shimura Var

But $\exists$ diff apparate in [FS] to show finiteness.