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Stack of vector bundles on the curve

E nonarch local field $\quad \mathcal{O}_E \ni \pi$

$\mathbb{F}_q \quad \overline{\mathbb{F}_q}$

$S \in \mathrm{Perf}_{\mathbb{F}_q} \rightsquigarrow X_S = X_{S,E}$

relative FF curve

Def'n Bun_n = the (pre) stack on $\mathrm{Perf}_{\mathbb{F}_q}$

$S \longmapsto \{$ rk n vector bundle on X_S $\}$

prop (1) Bun_n is a v-stack

(2) On $\mathrm{Perfd} = \{$ perf'd spaces $/ \mathbb{Z}_p \}$

have v-descent for vect bundles

proof 2) [Berkeley Lem 17.1.8]

know analytic descent by Kedlaya-Liu

have to prove

If $Y = \mathrm{Spa}(S, S^+) \longrightarrow X = \mathrm{Spa}(R, R^+)$

v-cover of affinoid perf'd spaces

($|Y| \to |X|$ surj)

then $\mathrm{Proj}(R) \simeq \{ N \in \mathrm{Proj}(S) + \alpha : N \hat{\otimes}_R S$

$\cong S \hat{\otimes}_R N$

of finite proj $S \hat{\otimes}_R S$-mod

satisfying cocycle condition

over $S \hat{\otimes}_R S \hat{\otimes}_R S$ $\}$

equiv of categories

already know fully faithfulness :

have right adj :

$(N, \alpha) \longmapsto \mathrm{eq}(N \rightrightarrows N \hat{\otimes}_R S)$

unit of adjunction

$$M \longmapsto \mathrm{eq}\,(M \hat{\otimes}_R S \rightrightarrows M \hat{\otimes}_R S \hat{\otimes}_R S)$$

is an isomorphism

(as structure sheaf is a v-sheaf)

only need to see effecitivity of descent data

Step 1 $R = K$ perfectoid field

may assume S (which is a K-Banach algebra) is topologically compactly generated

(Use: everything commutes with "countable-filtered colimit"

$\Rightarrow$ S is a free as a K-Banach space

In particular, $- \hat{\otimes}_K S$ exact conservative

Let (N, α) descent data

$$M := \mathrm{eq}(N \rightrightarrows N \hat{\otimes}_R S)$$

want $\quad M \hat{\otimes}_K S \xrightarrow{\sim} N$

But $\quad M \hat{\otimes}_K S = \mathrm{eq}(N \hat{\otimes}_K S \rightrightarrows N \hat{\otimes}_K S \hat{\otimes}_K S)$

$$\cong N$$

as

$$0 \to N \to N \hat{\otimes}_K S \to N \hat{\otimes}_K S \hat{\otimes}_K S \to \cdots$$

always exact

(as it admits contracting homology)

General Case $\quad$ Back to $R \to S$ general

$\forall x \in X$ with complete residue field $K(x)$

$\rightsquigarrow \quad K(x) \longrightarrow S \hat{\otimes}_R K(x)$ by base change

$\rightsquigarrow \quad$ descent

In particular, given any descent datum (N, α), $N \hat{\otimes}_S (S \hat{\otimes}_R K(x))$

is finite free and has α-invariant basis

choose a lifting $\xrightarrow[\text{spread out}]{}$ $\exists$ rational open $U \subseteq X$ $x \in U$

$N \hat{\otimes}_S (S \hat{\otimes}_R \mathcal{O}_X(U))$ finite free

and α given by a matrix

$\equiv 1 \mod \varpi$ for $\varpi \in R^+$

pseudo unif

enough to descent over $\mathcal{U}$ (by analytic descent)

so WLOG $U = X$

$$(N, \alpha) = (S^n, \alpha \in GL_n(S \hat{\otimes}_R S))$$

In fact, $\alpha \in GL_n(S^+ \hat{\otimes}_{R^+} S^+)$ $\alpha \equiv 1$ (ϖ)

Claim Can change basis s.t $\alpha = 1$

pf : sucessive approximation

Use : $\left(\frac{\alpha - 1}{\varpi} \bmod \varpi\right) \in M_n(S^+ \hat{\otimes}_{R^+} S^+ / \varpi)$

is an <u>additive</u> cocycle

But $H^1_v(X, \mathcal{O}^+/\varpi) \overset{\uparrow}{=} 0$

almost i.e killed by ϖ^{ε} $\forall \varepsilon > 0$

$\rightsquigarrow$ Can change to make

$\alpha \equiv 1 \bmod \varpi^{2-\varepsilon}$ $\forall \varepsilon > 0$

$\rightsquigarrow$ $\varpi^{3-2\varepsilon}$, ... $\Rightarrow$ $\alpha = 1$

1) Bun_n is a v-stack

$$S \longmapsto VB(X_S)$$

Use : $X_S \times_{\mathrm{Spa}\,E} \mathrm{Spa}\,E_\infty$ is perfectoid

v-cover to v-cover

2) $\rightsquigarrow$ descent VB on $X_S \times_E E_\infty$

Then descent along E_∞ / E by argument from Case 1 .

Q : Can one descent perfect complex ?

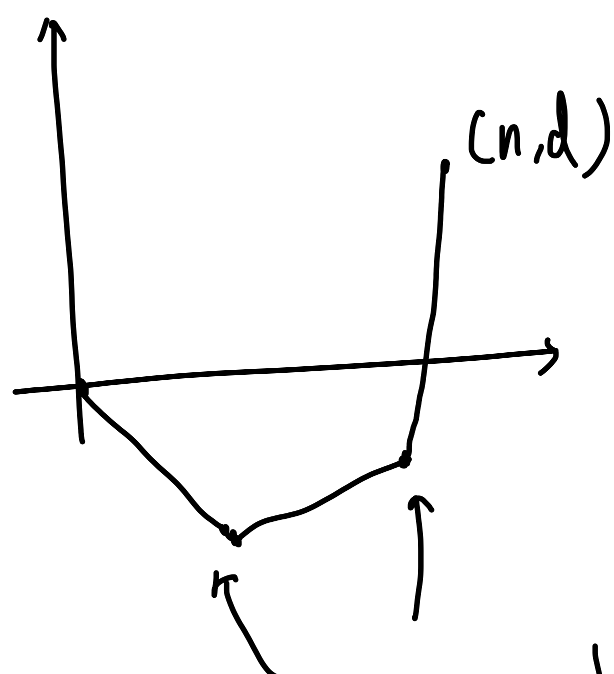
Rek In " Proj of Witt vector aff Gr "

Similar v-descent for VB & perf complexes are known on perfect schemes

Bun_n

$|\mathrm{Bun}_n|$ countable many pts

given by Newton polygon of width n



break pts have integral coordinates

let $B(GL_n)$ = set of such Newton polygons

$= \{n\text{-dim'l isocrystals}\}/\cong$

$\rightsquigarrow$ bijection $|\mathrm{Bun}_n| = B(GL_n)$

$|\mathcal{U}| \subset |\mathrm{Bun}_n|$
open if $\exists\, \mathcal{U} \overset{\text{open}}{\hookrightarrow} \mathrm{Bun}_n$

topological space $\rightsquigarrow$?

i.e if $X \twoheadrightarrow \mathrm{Bun}_n$ v-cover by perf'd

$$Y \twoheadrightarrow X \times_{\mathrm{Bun}_X} X$$

then $|\mathrm{Bun}_n| = |X| / |Y|$

<u>Question</u> on $B(GL_n) = ?$

partial order on $B(GL_n)$

$P \geq P'$ if P lies above P'
and have same end pts

$\rightsquigarrow$ topology on $B(GL_n)$:

$\mathcal{U} \subset B(GL_n)$ open

if $\forall\, P \in \mathcal{U}$, $\underline{P' \geq P}$ also $P' \in \mathcal{U}$

in particular, end(P') = end(P)

($\{b\}^c$ is open $\Leftrightarrow$ $\forall p \neq b$ $p' \geq p$ also $p' \neq b$ i.e. no $p \neq b$ s.t. $b \geq p$

<u>Thm</u> (Kedlaya-Liu, last time) (will be used in the course)

$|\mathrm{Bun}_n| \longrightarrow B(GL_n)$ continuous

Thm (Hansen et. al 2017)

$|\mathrm{Bun}_n| \xrightarrow{\sim} B(GL_n)$ homeomorphism

Rek $\forall$ general G, announced by Viehman

Picture: $|\mathrm{Bun}_1| = |\mathrm{Pic}| \cong \mathbb{Z}$

discrete !

$\vdots$

$\mathcal{O}(1)$

$\mathcal{O}$

$\mathcal{O}(-1)$

$\vdots$

$|\mathrm{Bun}_2|$

connected components

$d = 1$ $\quad \mathcal{O}(\frac{1}{2})$

$d = 0$ $\quad \mathcal{O}^2 \quad \mathcal{O}(-1) \oplus \mathcal{O}(1) \quad \mathcal{O}(-2) \oplus \mathcal{O}(2) \quad \cdots$

$d = -1$ $\quad \mathcal{O}(-\frac{1}{2})$

$d = -2$ $\quad \mathcal{O}(-1)^2 \quad \mathcal{O}(-2) \oplus \mathcal{O}$

$- \otimes_{\mathcal{O}} \mathcal{O}(1)$ makes the picture 2-periodic

$|\mathrm{Bun}_3|$

codim $= \langle 2\rho, \nu(b)\rangle$

there is a formula for codim by HN polygon

$d=0$

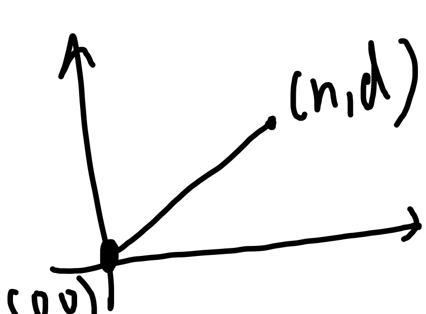
$$\begin{tikzcd} & \mathcal{O}(-\tfrac{1}{2})\oplus\mathcal{O}(1) \arrow[dr] & \\ \mathcal{O}^3 \arrow[ur] \arrow[dr] & & \mathcal{O}(-1)\oplus\mathcal{O}\oplus\mathcal{O}(1) \\ & \mathcal{O}(-1)\oplus\mathcal{O}(\tfrac{1}{2}) \arrow[ur] & \end{tikzcd}$$

Similar to classical story of Vect_n on $\mathbb{P}^1$

(but here only simple line bundles)

Obs $\pi_0 \mathrm{Bun}_n \cong \mathbb{Z}$ deg map

each connected comp has a unique semi-stable pt. (which is open): basic b = straight line



$$\mathcal{O}\left(\frac{d}{n}\right) := \mathcal{O}\left(\frac{a}{b}\right)^c$$

$d = ac \quad n = bc \quad (a,b)=1$

$\forall\ b \in B(GL_n)$ let

$Bun^b \subseteq Bun_n$ the preimage of b

<u>locally closed stratum</u>

1) $b = \mathcal{O}^n$

Cor of Thm of Kedlaya-Liu

$$\left\{ \begin{matrix} \text{semi-stable slope } 0 \text{ (pointwise)} \\ \text{VB on } X_S \end{matrix} \right\} \quad \forall\ S \in Perf_{\mathbb{F}_q}$$

$$\cong \left\{ \underline{E}\text{-local systems on } S \right\}$$

$$\mathcal{E} \longmapsto BC(\mathcal{E})$$

pf:

$$\mathbb{L} \otimes_{\underline{E}} \mathcal{O}_{X_S} \longleftarrow\!\!| \ \mathbb{L}$$

(pro-étale sheafication)

pro-étale descent

+ any $\mathcal{E}$ is pro-étale locally trivial

Cor $\quad \mathrm{Bun}_n^{\mathcal{O}} \cong \underbrace{[*/GL_n(E)]}$

$\uparrow$ classifying rk n $\underline{E}$-local systems

(So $\mathrm{Vect}(\mathcal{X}) \to \mathrm{Vect}(X)$ is $[*/GL_n(\mathcal{O})] \to [*/GL_n(E)]$)

Note: T any top space

$\rightsquigarrow \underline{T} : S \longmapsto \mathrm{Cont}(|S|, T)$

v-sheaf

$\rightsquigarrow$ open immersion ☆

$$[*/\underline{GL_n(E)}] \overset{j}{\hookrightarrow} \mathrm{Bun}_n$$

In particular

$$\{\text{reps of } GL_n(E)\} = \{\text{sheaves on } [*/GL_n(E)]\} \text{ (pro-étale) ☆}$$

$$\Big\downarrow j_!$$

$$\{\text{sheaves on } \mathrm{Bun}_n\}$$

2) semi-stable pts :

$\forall$ semi-stable bundle

$$\mathcal{O}\left(\frac{d}{n}\right) = \mathcal{O}\left(\frac{a}{b}\right)^{\oplus c}$$

$$\boxed{\underline{\mathrm{Aut}}\left(\mathcal{O}\left(\frac{d}{n}\right)\right) = \underline{GL_c(D_{a/b})}}$$

$D_{\frac{a}{b}} = \mathrm{End}\left(\mathcal{O}\left(\frac{a}{b}\right)\right)$ is the central div alg of E of Hasse inv $\frac{a}{b}$

$$\rightsquigarrow \quad \mathrm{Bun}_n^b \cong [\,*\,/\,\underline{GL_c(D_{a/b})}\,]$$

(E-pts) of inner forms of GL_n/E

$\rightsquigarrow$ $\{$reps of $GL_c(D_{\frac{a}{b}})\}$

$$\downarrow j_{b!}$$

$\{$sheaves on $Bun_n\}$

So Bun_n sees all reps of all inner forms of $GL_n(E)$

$\underbrace{\text{locally profinite}}$ gp

3) Non-semistable pts :

<u>example</u> Bun_2 , $b \simeq \mathcal{O} \oplus \mathcal{O}(1)$

always $Bun_n^b = [* / \underline{Aut}(\mathcal{E}_b)]$

$\underline{Aut}(\mathcal{E}_b)$ v-sheaf $S \longmapsto Aut(\mathcal{E}_b|_{X_S})$

$$\underline{\mathrm{Aut}}(\mathcal{O} \oplus \mathcal{O}(1)) = \begin{pmatrix} \underline{E}^{\times} & BC(\mathcal{O}(1)) \\ & \underline{E}^{\times} \end{pmatrix}$$

Recall $BC(\mathcal{O}(1))$ perfectoid open unit disc $\dim = 1 > 0$

$$1 \to BC(\mathcal{O}(1)) \to \underline{\mathrm{Aut}}(\mathcal{O} \oplus \mathcal{O}(1)) \to \underline{E}^{\times} \times \underline{E}^{\times} \to 1$$

1-dim, connected

<u>In general</u> $\mathcal{E}_b = \bigoplus_{\lambda \in \mathbb{Q}} \underbrace{\mathcal{O}(\lambda)^{n_\lambda}}_{\mathcal{E}_b^\lambda}$

$$\underline{\mathrm{Aut}}(\mathcal{E}_b) = \begin{pmatrix} \underline{\mathrm{Aut}}(\mathcal{E}_b^{\lambda_1}) & & & \\ & \underline{\mathrm{Aut}}(\mathcal{E}_b^{\lambda_2}) & & \\ & & \ddots & \\ & & & \underline{\mathrm{Aut}}(\mathcal{E}_b^{\lambda_m}) \end{pmatrix}$$

$\Rightarrow \quad 1 \longrightarrow$ "unipotent" $\longrightarrow \underline{\mathrm{Aut}}(\mathcal{E}_b)$

$\longrightarrow$ locally profinite gp $\to 1$

$\parallel$

$\mathrm{Aut}(V_b)$

$\parallel$

$J_b(E)$

extension of pos dim BC spaces

Kottwitz

V_b = isocrystal corresponding to b

(E-valued pts) of (inner form of a Levi of GL_n)

Note unipotent gps can't act on ℓ-adic sheaves ($\ell \neq p$)

$\rightsquigarrow$ $\{\ell\text{-adic sheaves on } \mathrm{Bun}_n^b\}$

$$\|$$

$\{\text{rep's of } J_b(E)\}$

$\rightsquigarrow$ $\{\ell\text{-adic sheaves on } \mathrm{Bun}_n\}$

$=$ semi-orthongonal decomposition by

from $\{\ell\text{-adic sheaves on } \mathrm{Bun}_n^b\}$

$$\|$$

$\{\text{rep's of } J_b(E)\}$

Q: How do different strata interact?

Ex $n=2$

closely to be smooth

a map $(\mathbb{P}^1_E)^\diamond / GL_2(E) \longrightarrow \mathrm{Bun}_2$:

Given $S^\# / E$ + $\mathcal{O}^2_{S^\#} \twoheadrightarrow L$

$(\cong$ a map $S \longrightarrow (\mathbb{P}^1_E)^\diamond)$

can build

$$\mathcal{E}(L) = \mathrm{Ker}(\mathcal{O}^2_{X_S} \twoheadrightarrow \mathcal{O}^2_{S^\#} \twoheadrightarrow L)$$

$S^\# \hookrightarrow X_S$ closed Cartier div

$\rightsquigarrow$ rk 2 VB on X_S

i.e $S \longrightarrow \mathrm{Bun}_2$

Prop The image lands in

$$
\begin{tikzcd}
\mathrm{Bun}_2^{\mathcal{O}(-\frac{1}{2})} & \sqcup & \mathrm{Bun}_2^{\mathcal{O}\oplus\mathcal{O}(-1)} \\
(\underline{\mathcal{O}}^2)^{\diamond} \arrow[u] & \sqcup & \mathbb{P}^1(E) \arrow[u] \\
\mathbb{P}^1_E \setminus \mathbb{P}^1(E) \arrow[u, equal]
\end{tikzcd}
$$

giving the above stratification of $\mathbb{P}^1_E$

<u>proof</u> $\mathcal{E}(L)$ neccessarily rk 2, deg -1

$\Rightarrow$ $\mathcal{E}(L) \cong \mathcal{O}(-\frac{1}{2})$ or $\mathcal{O}(-i-1) \oplus \mathcal{O}(i)$, $i \geq 0$

$$
\left(
\begin{tikzcd}
\mathcal{E}(L) \arrow[d, hook] \\
\mathcal{O}^2
\end{tikzcd}
\right) \Rightarrow i = 0
$$

If $\mathcal{E}(L) \simeq \mathcal{O}(-1) \oplus \mathcal{O}$ then

$$\underbrace{\mathcal{O} \hookrightarrow}\ \mathcal{E}(L) \hookrightarrow \mathcal{O}^2 \twoheadrightarrow \mathcal{O}^2_{S^\#} \twoheadrightarrow L$$

is zero

$$\hat{=} \quad E \xrightarrow{\neq 0} E^2 \to \mathbb{C}^2 \twoheadrightarrow L$$

is zero

$\Rightarrow$ L must be E-rational

so pt of $\mathbb{P}^1(E)$

✓

Conversely. the points lie in $\mathbb{P}^1(E)$

then up to $GL_2(E)$ action

Q:

Q: dim of Bun_n^b

(open question for dim of dimension)

$\dim \operatorname{Aut}(\mathcal{E}_b) = \langle 2\rho, \nu_b \rangle$

(no definition)

Q: $\overline{\mathbb{Q}}_\ell$, $\mathbb{Z}_\ell$, $\mathbb{F}_\ell$, $\mathbb{Z}/\ell^n$

coefficients

$\ell \neq p$

sheaves

Q: pullback $\vee$ $(\mathbb{P}^1)^\diamond \longrightarrow Bun_2$?

along

e.g. $\mathbb{P}^1_E \setminus \mathbb{P}^1(E) \longrightarrow [*/D^\times_{\frac{1}{2}}]$

is rep of $D^\times_{\frac{1}{2}}$ $\rightsquigarrow$ the local system $\mathcal{F}_\rho$

on Drinfeld half-plane

Q: v-descent along $E \to E_\infty$

$E_\infty \hat{\otimes}_E E_\infty$ is not uniform!