11/27

Families of vector bundles

Isocrystal v.s Vect bundles

moduli of ell curves

$\mathcal{M}_{ell} / \mathbb{Z}$ DM stack

$\mathcal{M}_{ell, \mathbb{F}_p} = \mathcal{M}^{ord}_{ell, \mathbb{F}_p} \cup \mathcal{M}^{ss}_{ell, \mathbb{F}_p}$

$\underline{\text{open}}$ $\underline{\text{closed, finite}}$

E/k ell curve char $k = p$ $(k = \overline{\mathbb{F}_p})$

$H^1_{crys}(E/W(k))[\frac{1}{p}] \in \mathrm{Iso}_{\mathbb{Q}_p}$

$\circlearrowright \phi$

is $\left(\check{\mathbb{Q}}_p^{2}, \begin{pmatrix} p & \\ & 1 \end{pmatrix} \sigma \right)$

slope $0, 1$ ordinary

or $\left(\check{\mathbb{Q}}_p^{2}, \begin{pmatrix} & 1 \\ p & \end{pmatrix} \sigma \right)$

slope $1/2$ supersingular

$\Rightarrow$ $\mathcal{M}_{ell, \overline{\mathbb{F}}_p}$ has family of isocrystals degenerating from ord to supersing

picture reversed when studying Vect on FF curve

$$\mathcal{M}_{ell, \overline{\mathbb{F}}_p} \subseteq \hat{\mathcal{M}}_{ell, \hat{\mathbb{Z}}_p} \supseteq \mathcal{M}^{ad}_{ell, \mathbb{C}_p}$$

formal scheme $/ \hat{\mathbb{Z}}_p = \mathcal{O}_{\mathbb{C}_p}$

adic generic fiber

$$Sp : |\mathcal{M}^{ad}_{ell, \mathbb{C}_p}| \longrightarrow |\mathcal{M}_{ell, \mathbb{F}_p}|$$

continuous

$\rightsquigarrow$ stratification on $\mathcal{M}^{ad}_{ell,\mathbb{C}_p}$ by pull back

$$\mathcal{M}_{ell, \mathbb{C}_p, p^\infty} \sim \varprojlim_m \mathcal{M}^{ad}_{ell, \mathbb{C}_p, p^m}$$

param isom $E[p^m] \cong (\mathbb{Z}/p^m)^2$

exists as perfectoid space

full isom

$E[p^\infty] \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^2$

$$\pi_{HT} : \mathcal{M}_{ell, \mathbb{C}_p, p^\infty} \longrightarrow \mathbb{P}^1_{\mathbb{C}_p}$$

by HT exact seq (rational will be exact)

$$0 \longrightarrow (\mathrm{Lie}\, E^*)^* \longrightarrow \underbrace{T_p E \otimes \mathbb{C}_p}_{\simeq \mathbb{C}_p^2} \longrightarrow \mathrm{Lie}\, E \longrightarrow 0$$

Prop E has ord red $\Longleftrightarrow$ HT filtration is rational

pf: ord $\Rightarrow$ $\bar{E}[p^\infty] \simeq \mu_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p$ lifts uniquely to E

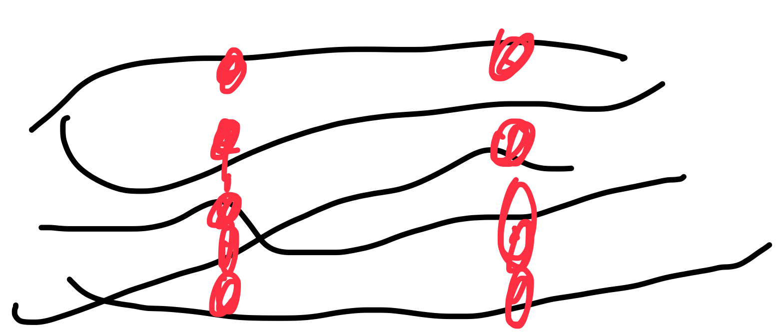
essentially

only true on rk 1 pts (because π_{HT} is cont)

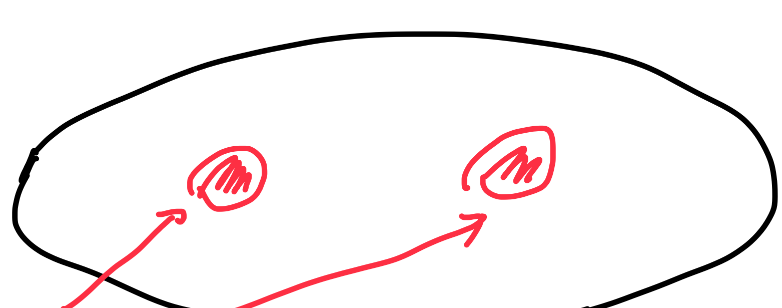
Q: Integrally ?

$$\mathcal{M}^{ord}_{ell,\mathbb{C}_p,p^\infty} = \pi_{HT}^{-1}(\mathbb{P}^1(\mathbb{Q}_p))$$

$\mathbb{P}^1(\mathbb{Q}_p) \subseteq \mathbb{P}^1_{\mathbb{C}_p}$ closed

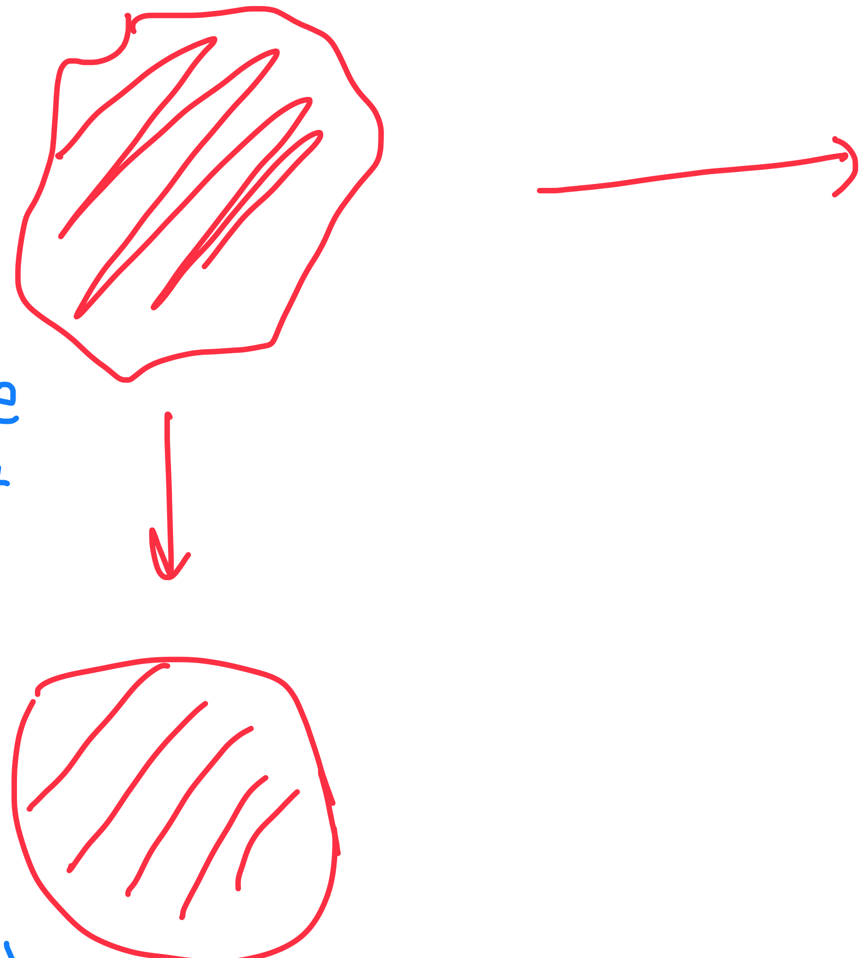


$\mathcal{M}_{ell,\mathbb{C}_p,p^\infty}$



disc of ss reduction

$\mathcal{M}_{ell,\mathbb{C}_p}$



Lubin–Tate tower at ∞-level

||

Drinfeld tower at ∞-level

$$\Omega^2 = \mathbb{P}^1_{\mathbb{C}_p} \setminus \mathbb{P}^1(\mathbb{Q}_p)$$

go to boundary of disc

$\cong$ go to boundary of Ω^2 $= \mathbb{P}^1(\mathbb{Q}_p)$

then rk 2 boundary pt maps to $\mathbb{P}^1(\mathbb{Q}_p)$ (spetral map is generalizing)

in the language of Vect on FF curve

$\mathbb{P}^1_{\mathbb{C}_p}$ parametrizes modification of the trivial rk 2 vect

$$\mathcal{O}^2_{X^b_{\mathbb{C}_p}} \;/\; \underbrace{X_{\mathbb{C}_p^b,\mathbb{Q}_p}}_{\text{FF curve}} \xleftarrow[\deg = 1]{} \operatorname{Spa} \mathbb{C}_p$$

i.e $x \in \mathbb{P}^1(\mathbb{C}_p) \rightsquigarrow \mathbb{C}_p^2 \twoheadrightarrow L_x$

Choosing a varying line at this pt

get

$$0 \to \mathcal{E}(L) \to \mathcal{O}^2 \to L \to 0$$

stalk at $\operatorname{Spa}\mathbb{C}_p$ $\rightsquigarrow$ $\mathbb{C}_p^2 \to L$ given by x

on Ω^2, $\mathcal{E}(L) \simeq \mathcal{O}(-\frac{1}{2})$

on $\mathbb{P}^1(\mathbb{Q}_p)$, $\mathcal{E}(L) \simeq \mathcal{O}(-1) \oplus \mathcal{O}$

E nonarch local field

$\cup\!|$ $\qquad \mathbb{F}_q$, $\overline{\mathbb{F}_q}$

$\mathcal{O}_E \ni \pi$

$S \in \mathrm{Perf}_{\mathbb{F}_q}$ perfectoid space

$\rightsquigarrow$ FF curve $X_S = X_{S,E}$

$\mathcal{E}$ vect bundle on X_S

$\forall$ each geo pt $\bar{s} = \mathrm{Spa}(C, C^+) \longrightarrow S$

(rather the strict henselization)

because of generalization

is trivial

VB on generic pt

Q: what is the fraction field of FF curve

$\rightsquigarrow$ $\mathcal{E}_{\bar{s}} \,/\, X_{\bar{s}}$

Note: $VB(X_{\bar{s}}) \cong VB(X_{\mathrm{Spa}(C, \mathcal{O}_C)})$

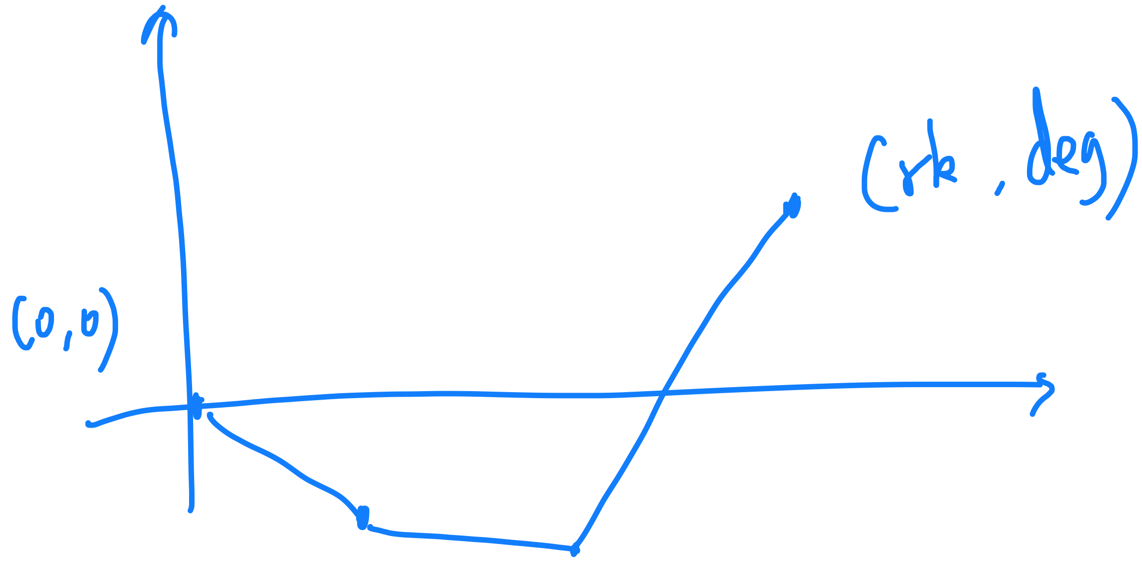
so can forget about C^+

classification $\rightsquigarrow$ $\mathcal{E}_{\bar{s}} \cong \bigoplus_{\lambda \in \mathbb{Q}} \underset{X_{\bar{s}}}{\mathcal{O}(\lambda)}^{n_x(\bar{s})}$

pf: VB on $\mathrm{Spa}(R, R^+)$ = projective R-module

so no R^+

$\rightsquigarrow$ Newton polygon



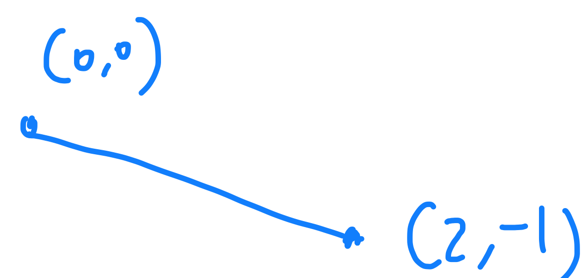
length $\lambda \cdot n_\lambda(\bar{s})$ $\forall$ slope λ

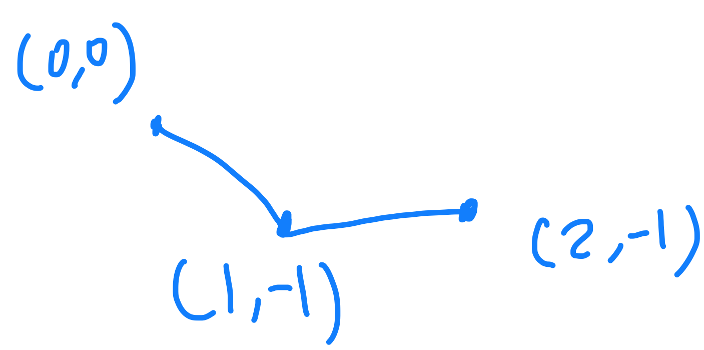
<u>Ex</u>

$\mathcal{O}(-\frac{1}{2})$ generic one

$\rightsquigarrow$ sp

$\mathcal{O}(-1) \oplus \mathcal{O}$ special one





Q: How does the Newton polygon vary?

<u>ordering</u>: $P \geq P'$ if $\begin{cases} P \text{ is above } P' \\ P, P' \text{ have same end pts} \end{cases}$

Thm (Kedlaya - Liu '15)

1) $\bar{s} \longmapsto NP(\mathcal{E}_{\bar{s}})$

is upper semi-continuous

(see the example above)
easy to remember

2) If Newton polygon is constant

then there is a global HN filtration

$$\mathcal{E}^{\geq \lambda} \subseteq \mathcal{E}$$

by strict vector bundle

$$\mathcal{E}^{\lambda} = \mathcal{E}^{\geq \lambda} \Big/ \bigcup_{\lambda' > \lambda} \mathcal{E}^{\geq \lambda'}$$

quotient still
vect bundle

is everywhere semi-stable slope of λ

Today: new proof using diamonds + v-descent

Key Projectivized Banach-Colmez spaces are **proper** ☆

Separated + proper maps

Def'n $f: \mathcal{F} \to \mathcal{G}$ map of small v-sheaves

1) f is **closed immersion**

if $\forall$ all strictly totally disconnected (to make sure this doesn't descent) Zariski closed and

X, $X \to \mathcal{G}$ the fiber product

$\mathcal{F} \times_{\mathcal{G}} X$ is a perfd space X'

and $X' \to X$ a (Zariski) closed imm

equiv: $\exists$ $Z \subset |\mathcal{G}|$ closed generalizing subset s.t $\mathcal{F} \subseteq \mathcal{G}$ is the subfunctor

$$\{ X \to \mathcal{G} \text{ s.t } |X| \to |\mathcal{G}| \text{ factor over } Z \}$$

2) f is separated if Δ_f is a closed immersion

3) proper $\overset{\text{def}}{\Longleftrightarrow}$ separated + universal closed + quasi-compact

Note: no finite type assumption

$\exists$ valuative criterion :

Prop f separated (proper) $\iff$ quasi-separated (qcqs) + uniqueness in

$$\begin{tikzcd}
\mathrm{Spa}(R, R^\circ) \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\
\mathrm{Spa}(R, R^+) \arrow[r] \arrow[ur, dashed] & \mathcal{G}
\end{tikzcd}$$

$\forall$ all affinoid perf $\mathrm{Spa}(R, R^+)$

In fact, enough to check for

$$(R, R^+) = (C, C^+)$$

complete alg closed

valuation subring

Def'n partially proper = " proper without quasi-compact "

i.e separated + $\exists$

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\
\bullet \arrow[r] \arrow[ur, dashed] & \mathcal{G}
\end{tikzcd}$$

<u>prop</u> Let $f: \mathcal{F} \to \mathcal{G}$ be quasi-cpt

Then f is surjective as a map of v-sheaves iff $|f|: |\mathcal{F}| \to |\mathcal{G}|$ is surjective

pf: reduce to reps case $\mathcal{F} = X$, $\mathcal{G} = Y$

But then $X \to Y$ is a v-cover $\square$

Projectivized Banach-Colmez Spaces

<u>prop</u> $\forall\ S \in \mathrm{Perf}_{\mathbb{F}_q}$ $\mathcal{E} \in \mathrm{VB}(X_S)$

$$BC(\mathcal{E}) : T/S \mapsto H^0(X_T, \mathcal{E}|_{X_T})$$

is a <u>locally spatial diamond</u> <u>partially proper $/S$</u>

The proj BC space is

$$(BC(\mathcal{E}) \setminus \{0\})/\underline{E}^{\times}$$

is a local spatial diamond, proper /S

quasi-cpt !

Pf: classically $\mathcal{E} \hookrightarrow \mathcal{O}^n$ $BC \hookrightarrow \mathbb{A}^n$ ✓

$\mathbb{P}(BC) \hookrightarrow \mathbb{P}^{n-1}$

Now $\mathcal{O}(1)$ ample $\Rightarrow$ $\exists$ surj

$\mathcal{O}(-n)^N \twoheadrightarrow \mathcal{E}^\vee$ $n, N \gg 0$

so $\mathcal{E} \hookrightarrow \mathcal{O}(n)^N$

$\leadsto$ closed immersion

$BC(\mathcal{E}) \hookrightarrow BC(\mathcal{O}(n)^N)$

reduce to $\mathcal{E} = \mathcal{O}(n)^N$

partially proper: clear as VB doesn't depend on R^+

then use $0 \to \mathcal{O}(n-1) \to \mathcal{O}(n) \to \mathcal{O}_{S^\#} \to 0$

to do induction

$$\rightsquigarrow \quad 0 \to BC(\mathcal{O}(n-1)^N) \to BC(\mathcal{O}(n)^N)$$

$$\to (\mathbb{A}^N_{S^\#})^\diamond \to 0$$

qs. locally spatial diamond

<u>hard part</u>: $(BC(\mathcal{E}) \setminus \{0\})/\underline{E}^\times$ is quasi-compact

assume (S is qcqs)

$$\Updownarrow \quad \mathcal{O}_E^\times \text{ cpt}$$

$$BC(\mathcal{E}) \setminus 0 \,/\, \pi^{\mathbb{Z}} \text{ is qc}$$

$$\Updownarrow$$

$$(|BC(\mathcal{E})| \setminus \{0\})/\pi^{\mathbb{Z}} \text{ is qc}$$

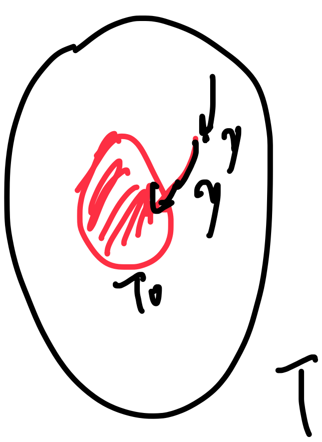
it's a Banach space

Idea: "contracting" automorphism of local spectral spaces

$\gamma \curvearrowright T \longleftrightarrow T_0 = T^{\gamma}$ closed

assume T_0 is spetral

Assume : • " T looks like analytic adic space "



• $n \to \infty$, the action of γ^n contracts towards T_0

• $n \to -\infty$ . . . . . γ^n on $T \setminus T_0$ diverges

Outpt : $\begin{cases} \gamma \text{ acts freely on } T \setminus T_0 \\ T / \gamma^{\mathbb{Z}} \text{ is spectral i.e qcqs} \end{cases}$

pf : pointed set topology on spectral space $\square$

Back to <u>Kedlaya-Liu</u>

$S \in \mathrm{Perf}_{\mathbb{F}_q}$, $\mathcal{E} \in VB(X_S)$

1) $NP(\mathcal{E}) : s \longmapsto NP(\mathcal{E}_s)$

is semi-continuous

Note: $NP(\mathcal{E}_{\bar{s}})$ = convex hull of all pts (i, d_i) $\forall i = 0, \dots, \mathrm{rk}\,\mathcal{E}$

s.t. $\exists \neq 0$ section of

$\Lambda^i \mathcal{E}(-d_i)$

$\rightsquigarrow$ enough to show:

the locus of non-zero sections is closed in S

But this is precisely

Image of $|BC(\mathcal{E}) \setminus \{0\} / \underline{E}^\times| \to S$

But proper $\Rightarrow$ image closed

2) If $NP(\mathcal{E})$ constant, then $\exists$ global HN filtration, and pro-étale locally

$$\mathcal{E} = \bigoplus_{\lambda \in \mathbb{Q}} \mathcal{O}(\lambda)^{n_\lambda}$$

Claim enough to show it v-locally

indeed then HN filtration exist v-locally, so descend (by v-descent of Vect)

Isom $\quad \mathcal{E}^{\lambda} \cong \mathcal{O}(\lambda)^{n_\lambda}$

is a v-torsor under $\underline{GL_{n_\lambda}(D_\lambda)}$

thus a pro-étale torsor

$\rightsquigarrow$ can find $\mathcal{E}^{\lambda} \cong \mathcal{O}(\lambda)^{n_\lambda}$ pro-étale locally split HN filtration :

use $H^1(X_S, \mathcal{O}(\lambda)) = 0 \quad \forall \lambda > 0$

any S affinoid

proof of the claim :

$\lambda :=$ max slope of $\mathcal{E}$

want to find a fiberwise non-zero

$\mathcal{O}_{X_S}(\lambda) \longrightarrow \mathcal{E}$ after a v-cover

Then $0 \longrightarrow \mathcal{O}(\lambda) \to \mathcal{E} \longrightarrow \overline{\mathcal{E}} \longrightarrow 0$

NP still constant

win by induction

But let $\mathcal{E}' = \mathrm{Hom}(\mathcal{O}(\lambda), \mathcal{E})$

$$BC(\mathcal{E}')\setminus\{0\} \xrightarrow[\text{surj}]{} BC(\mathcal{E}')\setminus\{0\}/\underline{E}^{\times} \longrightarrow S$$

proper surj on geometric pts

is itself a v-cover

$\Rightarrow$ surjective as v-sheaves

so it's a v-cover

Tautologically on $BC(\mathcal{E}')\setminus\{0\}$

we have fiberwise non-zero map $\mathcal{O}(\lambda) \to \mathcal{E}$

n

Q: pts of Proj BC space ?

$$BC(\mathcal{O}(1)\setminus\{0\})/\underline{E}^{\times}$$

perfectoid punctured open unit disc

on $\mathrm{Perf}_{\mathbb{F}_q}$: $BC(\mathcal{O}(1))\setminus\{0\} \cong (\mathrm{Spa}\, \check{E}_{\infty}^{LT})^{\diamond}$

$$\left(BC(\mathcal{O}(1)) \setminus \{0\} \right) / \underline{E}^\times$$

$$\cong \left(\mathrm{Spa}\, \check{E}^{LT}_{\sigma} \right)^{\diamond} / \underline{E}^\times$$

$$\cong \left(\mathrm{Spa}\, E \right)^{\diamond} / \varphi^{\mathbb{Z}}$$

$$\cong \mathrm{Div}^1_X$$

$$BC(\mathcal{O}(d)) \setminus \{0\} / \underline{E}^\times \simeq \mathrm{Div}^d_X$$

Q:

hard to classify VB on general S

but study in moduli

if X stric tot disc

then

$$\{\text{closed } Z \subseteq X\} \cong \{\text{closed } S \subset \pi_0 X\}$$

profinite

Q: compute coh of $BC(\mathcal{O}(d))$

In principle

Q: closed subsets

Q: $\mathrm{Ext}^1(\mathcal{O},\mathcal{O})$ is zero v-locally

$$\star \quad H^1(X_{S},\mathcal{O}) = H^1_{\text{pro\'et}}(S,\underline{E})$$

$$0 \to \mathcal{O}_{X_S} \to \mathcal{O}_{X_S}(1) \to \mathcal{O}_{S^\#} \to 0$$

$$\hat{G}^{ad}(R^\#) \xrightarrow{\log \hat{G}} R^\#$$

pro-étale locally surjective

$$\hookrightarrow G^{ad}(R^\#) \nearrow_{\log G}$$

$H^*(\mathcal{M}^{ad,ss}_{ell,\infty,\mathbb{C}_p}) \circlearrowleft$ geometric LLC

minimal compactification is canonical

HT period map

Auto vect bundle extends to minimal compactification

(at ∞-level)