Families of vector bundles

Isocrystal v.s. Vect bundles

moduli of ell curves

\[ \mathcal{M}_\text{ell} / \mathbb{Z} \] DM stack

\[ \mathcal{M}_\text{ell}, \overline{\mathbb{F}_p} = \mathcal{M}_\text{ell}, \mathbb{F}_p \cup \mathcal{M}_\text{ell}, \overline{\mathbb{F}_p} \] open \hspace{1cm} closed, finite

\[ E / k \text{ ell curve} \]
\[ \text{char } k = p \hspace{1cm} (k = \overline{\mathbb{F}_p}) \]

\[ H^1_{\text{cris}} \left( \mathcal{E} / W(k) \right) \left[ \frac{1}{p} \right] \in \text{Isom} \mathbb{Q}_p \]
is \( (\mathbb{Q}_p^2, (p,1)^\sigma) \)

slope 0,1 ordinary

or \( (\mathbb{Q}_p^2, (p,1)^\sigma) \)

slope 1/2 supersingular

\[ \Rightarrow M_{ell, \overline{\mathbb{F}_p}} \text{ has family of isocrystals} \]

degenerating from ord to supersing

picture reversed when studying Vect on FF curve

\[ M_{ell, \overline{\mathbb{F}_p}} \leq M_{ell, \mathbb{Z}_p} \leq M_{ell, \mathbb{C}_p} \]

adic generic fiber

formal scheme

\[ / \mathbb{Z}_p = 0 \mathbb{C}_p \]
$Sp : \left| \text{Mell, } C_p \right| \longrightarrow \left| \text{Mell, } \overline{\mathbb{F}_p} \right|$

\[ \rightsquigarrow \text{Stratification on } \text{Mell, } C_p \text{ by pull back} \]

\[ \text{Mell, } C_p, p^\infty \sim \varprojlim_m \text{Mell, } C_p, p^m \]

\[ \text{full isom} \rightarrow \text{exists as perfectoid space} \]

$E[p^\infty] \cong (\mathbb{Q}_p / \mathbb{Z}_p)^2$

$E[p^m] \subseteq (\mathbb{Z}/p^m)^2$

$\pi_{HT} : \text{Mell, } C_p, p^\infty \longrightarrow \mathbb{P}^1_{C_p}$

by HT exact seq (Rational will be exact)

$0 \longrightarrow (\text{Lie } E^*)^* \longrightarrow T_p E \otimes C_p \longrightarrow \text{Lie } E$

$\cong C_p^2 \longrightarrow 0$
Prop: $E$ has ord red

$\iff$ HT filtration is rational

pf: ord $\Rightarrow E[\ell^\infty] \cong M_{\text{ord}} \times \mathbb{Q}_p / \mathbb{Z}_p$

lifts uniquely to $E$

essentially only true on $\text{HT}$ 1 pts (because $\pi_{\text{HT}}$ is cont)

$M_{\text{ord}} = \pi_{\text{HT}}^{-1}(\ell_1'(\mathbb{Q}_p))$

$\ell_1'(\mathbb{Q}_p) \subseteq \ell_1'_{\mathbb{Q}_p}$ closed

$M_{\text{ell}, C_\ell, p\infty}$

$\text{disc of ss reduction}$
Lubin-Tate tower at $\infty$-level

Drinfeld tower at $\infty$-level

go to boundary of disc

\[ \cong \text{go to boundary of } \Omega^2 \]

then $\text{rk } 2$ boundary pt maps to $\text{rk } 1 \Omega_p^1$ (spectral map is generalizing)

in the language of Vect on $\text{FF}$ curve

$\Omega_p^1$ parametrizes modification of the trivial $\text{rk } 2$ vect
Choosing a varying line at this pt

get

$\begin{align*}
0 \rightarrow E(L) \rightarrow O^2 \rightarrow L \rightarrow 0
\end{align*}$

on $\Omega^2, E(L) \simeq O(-\frac{1}{2})$

on $\mathbb{P}^1(C_p), E(L) \simeq O(-1) \oplus O$

---

$E$ nonarch local field

$U_1 \supset \mathcal{O}_E \equiv \pi$

$\mathcal{O}_E \equiv \pi$ 1\text{F}_q, \overline{1\text{F}}_q

$S \in \text{Perf}_{1\text{F}_q}$ perfectoid space
\[ S \text{ FF curve } X_S = X_{S,E} \]

\[ E \text{ vect bundle on } X_S \]

\[ \forall \text{ each geo pt } \bar{s} = \text{Spec}(C, C^+) \rightarrow S \]

(rather the strict henselization)

because of generalization

VB on generc pt

Q: what is the

fraction field of

FF curve

\[ \sim \quad E_{\bar{s}} / X_{\bar{s}} \]

Note:

\[ \text{VB}(X_{\bar{s}}) \cong \text{VB}(X_{\text{Spec}(C, \mathcal{O}_C)}) \]

so can forget about \( C^+ \)

classification

\[ E_{\bar{s}} \cong \bigoplus_{\lambda \in Q} \mathcal{O}(\lambda)^{n_{X_{\bar{s}}}(\lambda)} \]

\[ \sim \quad \text{Newton polygon} \]

\[ \text{VB} = \text{projective} \quad \text{on Spec}(\mathbb{R}, \mathbb{R}) \quad \text{R-module} \quad \text{so by } R^+ \]
Ex

\( O(-\frac{1}{2}) \) generic one
\[ \frac{1}{2} sp \]
\( O(-1) \oplus O \) special one

\( (0,0) \) \( (2,-1) \)

Q: How does the Newton polygon vary?

ordering: \( P \geq P' \) if \( \forall i \), \( P \) is above \( P' \)
\( P, P' \) have same end pts
Thm (Kedlaya–Liu '15)

1) \( s \xrightarrow{\text{is semi-continuous}} NP(E) \)

(see the example above)

easy to remember

2) If Newton polygon is constant

then there is a global HN filtration

\[ E^\lambda \subseteq E \]

by strict vector bundle

\[ E^\lambda = E_{\geq \lambda} \cup E_{< \lambda} \]

quotient still vect bundle

is everywhere semi-stable slope of \( \lambda \)

Today: new proof using diamonds + \( \nu \)-descent
Key

Projectivized Banach-Colmez spaces are proper.

Separated + proper maps → small

Defn. \( f : \mathcal{F} \to \mathcal{G} \) map of \( \nu \)-sheaves

1) \( f \) is closed immersion

if \( \forall \) all strictly totally disconnected

\[ X, X \to \mathcal{G} \] the fiber product

\[ \mathcal{F} \times_{\mathcal{G}} X \] is a perfed space \( X' \)

and \( X' \to X \) (Zariski) closed imm

equiv.: \( \exists Z \subset |G| \) closed generalizing subset
s.t. $F \subseteq G$ is the subfunctor

$$\begin{cases} x \rightarrow G \quad & \text{s.t. } |x| \rightarrow |G| \\ \text{factor over } \mathbb{Z} \end{cases}$$

2) $f$ is separated if $\Delta_f$ is a closed immersion

3) proper

\[
\xrightarrow{\text{def}} \quad \text{separated} \quad + \quad \text{universal closed} \quad + \quad \text{quasi-compact}
\]

Note: no finite type assumption

$\exists$ valuative criterion;
Prop \[ f \text{ separated } \iff \text{ quasi-separated (qcqs)} \]

(proper) + uniqueness in

\[
\begin{array}{ccc}
\text{Spa}(R, R^0) & \longrightarrow & \mathbb{F} \\
\downarrow & & \downarrow \\
\text{Spa}(R, R^+) & \longrightarrow & \mathbb{G}
\end{array}
\]

All affinoid perf \( \text{Spa}(R, R^+) \)

In fact, enough to check for

\[(R, R^+) = (C, C^+), \]

complete alg closed valuation subring

Def'n partially proper = "proper without quasi-compact"

i.e. separated + \[ \begin{array}{ccc}
\delta & \longrightarrow & \mathbb{E} \\
\delta & \longrightarrow & \mathbb{G}
\end{array} \]
Prop. Let $f: F \to G$ be quasi-compact. Then $f$ is surjective as a map of $v$-sheaves iff $\vert f \vert: \vert F \vert \to \vert G \vert$ is surjective.

Proof: Reduce to reps case $F \cong G$.

But then $X \to Y$ is a $v$-cover.

Projectivized Banach-Culmezd Spaces

Prop. \quad \forall S \in \text{Perf}_{F_\eta}, \quad \exists \in \text{VB}(X_S)

$BC(\exists): T/S \to H^0(X_T, \exists|_{X_T})$

is a locally spatial diamond partially proper $/S$.
The proj BC space is
\[ \frac{(BC(E) \setminus \{0\})}{\mathbb{E}^x} \]
is a local spatial diamond, proper $S$ quasi-opt!

pf: classically $E \hookrightarrow \mathbb{O}^n$
$BC \hookrightarrow \mathbb{A}^n$\checkmark
$IP(BC) \hookrightarrow IP^{n+1}$

Now $OCl$ ample $\Rightarrow \exists$ surj

$O(-n)^N \rightarrow E^v$
so $E \hookrightarrow O(c)^N$

closed immersion

$BC(E) \rightarrow BC(O(c)^N)$ reduce to $E = O(c)^N$
partially proper;

$VB$ doesn't depend on $R^+$

then use $0 \rightarrow O(n-1) \rightarrow O(n) \rightarrow O_S^* \rightarrow 0$
to do induction
\[ 0 \rightarrow BC(O(n-1)^N) \rightarrow BC(O(n)^N) \rightarrow (1A\mathcal{N}_{S\#})^0 \rightarrow 0 \]

q.s. locally spatial diamond

hard part: \((BC(E) \setminus \{0\}) / E^x\) is quasi-compact

assume \((S \text{ is q.c.g.s.})\)

\[ O^x_E \text{ cpc} \]

\[ BC(E) \setminus 0 / \pi^2 \text{ is qc} \]

it's a Banach space

\[ (|BC(E)| \setminus \{0\}) / \pi^2 \text{ is qc} \]

Idea: "contracting" automorphism of local spectral spaces
\[ \gamma \circ T \leftrightarrow T_0 = T \gamma \]

Assume: 
- \( T \) looks like analytic adic space
- \( n \to \infty \), the action of \( \gamma \)
  contracts towards \( T_0 \)
- \( n \to -\infty \), \( \ldots \ldots \gamma^n \) on \( T \setminus T_0 \) diverges

Output: \( \gamma \) acts freely on \( T \setminus T_0 \)
- \( T / \gamma \) is spectral i.e. qcqs

pf: pointed set topology on spectral space

\[ \square \]

Back to Kedlaya-Liu
\( S \in \text{Perf}_{\text{Eq}}, \ E \in \text{VB}(X_S) \)

1) \( NP(E) : S \rightarrow NP(E_S) \)

is semi-continuous

Note: \( NP(E_S) = \text{convex hull of} \)

all pts \((i, d_i) \) \( \forall i = 0, \ldots, \text{rk} E \)

s.t \( \exists \ E \neq 0 \) section of \( \bigwedge E (-d_i) \)

\( \bigwedge \) enough to show:

the locus of non-zero sections is closed in \( S \)

But this is precisely

Image of \( \left| \text{BC}(E) \setminus \{0\} \right| / E^x \rightarrow S \)
But proper $\implies$ image closed.

2) If $NP(E)$ constant, then $E$ global $HN$ filtration, and pro-étale locally

$E = \bigoplus_{\lambda \in EQ} O(\lambda)^{n_{\lambda}}$

Claim enough to show it $\nu$-locally

indeed then $HN$ filtration exist

$\nu$-locally, so descend (by $\nu$-descent of Vect)

Isom $E^{\lambda} \cong O(\lambda)^{n_{\lambda}}$

Is a $\nu$-torsor under $GL_{n_{\lambda}}(D_{\lambda})$

thus a pro-étale torsor

we can find $E^{\lambda} \cong O(\lambda)^{n_{\lambda}}$ pro-étale locally
split HN filtration:

use $H^1(X_S, \mathcal{O}(\lambda)) = 0$ for any $\lambda > 0$

proof of the claim:

$\lambda := \max \text{ slope of } \mathcal{E}$

want to find a fiberwise non-zero

$\mathcal{O}_{X_S}(\lambda) \to \mathcal{E}$ after a $\nu$-cover

Then $0 \to \mathcal{O}(\lambda) \to \mathcal{E} \to \overline{\mathcal{E}} \to 0$

NP still constant

win by induction

But let $\mathcal{E}' = \text{Hom}(\mathcal{O}(\lambda), \mathcal{E})$
\[ BC(E) \setminus \{0\} \xrightarrow{\text{surj}} BC(E) \setminus \{0\} / E^x \xrightarrow{\text{proper}} S \]

is itself a \( v \)-cover

"Tautologically on \( BC(C) \setminus \{0\} \)"

we have fibrewise non-zero map \( O(C) \to \mathbb{E} \)

\[ Q: \text{pts of Proj } BC \text{ space?} \]

\[ BC(C) \setminus \{0\} / E^x \]

\text{perfectoid punctured open unit disc}

on \( \text{Proj } F \): \( BC(C) \setminus \{0\} \cong (\text{Spa } E_{\infty}) \)
\[
(\mathcal{BC}(\mathcal{O}(1)) \setminus \{0\}) / E^x
\cong (\text{Spa } E_x^+)^0 / E^x
\cong (\text{Spa } E)^\circ / \mathfrak{m}_E
\cong \text{Div}^1_x
\]

\[
B \in \mathcal{O}(d)) \setminus \{0\} / E^x \cong \text{Div}^d_x
\]

Q:

- hard to classify $VB$ or $S$
- but study in moduli

if $X$ strict tot disc
then
\[
\{ \text{closed } Z \subseteq X \} \cong \{ \text{closed } S \subseteq \prod_0 X \} \quad \text{profinite}
\]

\[\hat{Q} : \text{compute } \chi \text{ of } BC \circ (c)\]

In principle

\[\hat{Q} : \text{closed subset}\]

\[\hat{Q} : \text{Ext}^1(0,0) \text{ is zero } \text{v-locally}\]

\[\star \quad H'(X_S, O) = H'_\text{proét}(S, E)\]

0 → O_{X_S} → O_{X_S}(1) → O_{\ast} \rightarrow 0

\[
\hat{\alpha}_\text{ad}(R^\#) \xrightarrow{\log \hat{\alpha}} R^\# \quad \text{pro-etale locally surjective}
\]

\[
\hat{\alpha}_\text{ad}(R^\#) \xrightarrow{\log \hat{\alpha}} R^\#
\]

\[
\hat{\alpha}_\text{ad}(R^\#) \xrightarrow{\log \hat{\alpha}} R^#
\]
\[ H^* (\text{Mell, } \infty, \mathcal{C}_p) \xrightarrow{\text{generic LLC}} \]

\text{minimal compactification is canonical}

\[ H^7 \text{ period map} \]

\text{Auto vell bundle extends to}

\[ \text{minimal compactification} \]

\[ \text{at } \infty \text{-level} \]