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$S/\mathbb{F}_q$ perfd space

Aim Introduce rel FF curve

$$X_{S,E} = Y_{S,E}/\phi^{\mathbb{Z}}$$

$$Y_{S,E} \text{ "}=\text{" } S \times \operatorname{Spa} E \text{ "}$$

$$Y_{S,E}^{\diamond} = S \times (\operatorname{Spa} E)^{\diamond}$$

$$\{\text{analytic adic spaces}/\mathbb{Z}_p\} \to \{\text{diamonds}\}$$
$$X \longmapsto X^{\diamond}$$

sending perfd space $\longmapsto$ perfd space in char p

$$X \longmapsto X^{\flat}$$

pro-étale local structure of perfectoid spaces

Def'n A perfd space X is

(strictly) totally disconnected if

it's qcqs (in fact affinoid)

and every étale cover splits (strictly tot disc)

open cover splits (tot disc)

Prop X is (strictly) tot disc

iff it's qcqs and all fibres of

$$X \longrightarrow \pi_0(X)$$

profinite set

are of form

$$\mathrm{Spa}(K, K^+)$$

where K is perfd field

(= complete nonarch non-discretely valued field s.t Φ surj on $\mathcal{O}_K/p$)

and $\mathfrak{m}_{\mathcal{O}_K} \subset K^+ \subset \mathcal{O}_K$

valuation subring

(Resp and K is alg closed for strictly tot disc)

Remark $K^+/\mathfrak{m}_{\mathcal{O}_K} \subseteq \mathcal{O}_K/\mathfrak{m}_{\mathcal{O}_K} = k$

$\leadsto$ $|\mathrm{Spa}(K,K^+)| = |\mathrm{Spec}(\underbrace{K^+/\mathfrak{m}_{\mathcal{O}_K}}_{\text{valuation ring}})|$

= totally ordered chain of specializations

generic pt $\simeq \mathrm{Spa}(K,\mathcal{O}_K) \hookrightarrow \mathrm{Spa}(K,K^+)$

it's the unique rk 1 generalization of any pt

Cor Assume $X = \mathrm{Spa}(R,R^+)$ tot disc

$f: Y = \mathrm{Spa}(S,S^+) \longrightarrow X$

any aff'd adic space over X

Then $R^+/\varpi \longrightarrow S^+/\varpi$ flat

$\forall$ any pseudo-unif ϖ of R

(and f is faithfully flat if $|f|$ surjective)

proof check it on connected component

Then $(R,R^+) = (K,K^+)$

($R^+ \to S^+$ may not be flat because $\lim$ colim may not comm)

So K^+ is a valuation ring

(over it flat $\Leftrightarrow$ torsion free)

$$\Rightarrow \quad S^+ \subset S = S^+[\tfrac{1}{\varpi}]$$

is flat over K^+

$$\Rightarrow \quad S^+/\varpi \text{ flat over } K^+/\varpi$$

For faithful flatness, use

$$|\mathrm{Spa}(K, K^+)| \cong |\mathrm{Spec}(K^+/\varpi)|$$

$\square$

This allows us to deduce v-descent from pro-étale descent + f.f descent

Def'n A diamond is a pro-étale sheaf Y on $\mathrm{Perf} = \{\text{perfd space}/\mathbb{F}_p\}$ that can be written as

$$Y = X/R$$

Analogue of algebraic spaces

where – X perfd space

– $R \subset X \times X$ equiv relation
represent by a perfd space
s.t $pr_i : R \to X$ pro-étale

Here, use the Yoneda embedding

$$\mathrm{Perf} \hookrightarrow \{\text{pro-étale sheaves on } \mathrm{Perf}\}$$
$$X \longmapsto \mathrm{Hom}(-, X)$$

proved by showing $\mathcal{O}_X$ is a sheaf

Facts • Category of diamonds
has all fibre products, cofiltered inv limits,
(all nonempty limits)

but no final object

Reason: the final obj would be $\mathrm{Spa}\,\mathbb{F}_p$
(not perfd, because not analytic!)

and can't add a top-nilpotent unit even after pro-étale covers

- If $f: Y \to X$ quasi-pro-étale map

then Y diamond $\Leftarrow$ X diamond

($\Rightarrow$ if f is surjective as maps between pro-étale sheaves)

"maps with profinite fibers" "pro-étale" locally

better notion to work locally

<u>Recall</u>

if $\forall$ all str. tot. disc. perf'd space X', $X' \to X$, the fibre prod

$f': Y' = Y \times_X X' \to X'$ is rep

in perf'd spaces, and pro-étale

$\boxed{\text{quasi-pro-étale} \not\Rightarrow \text{pro-étale} \quad \text{"ramified disc" last time}}$

- Y diamond $\Leftrightarrow$ $\exists$ surj quasi-pro-étale $X \to Y$, X perf'd space topological space $|Y| := |X| / |R|$

(independent of presentation)

Ex. Fix geom base pt $S = \mathrm{Spa}(C, \mathcal{O}_C)$

$$\mathrm{ProFin} \hookrightarrow \mathrm{Perf}/S$$

$$T = \lim_i T_i \longmapsto \lim_i (T_i \times \mathrm{Spa}(C, \mathcal{O}_C))$$

$$= \mathrm{Spa}(\mathrm{Cont}(T, C), \mathrm{Cont}(T, \mathcal{O}_C))$$

Recall Any compact Hausdorff space T is a quotient of profinite set = tot. disc. compact Hausdorff space

i.e $T = \tilde{T} / R$ $\quad R \subset \tilde{T} \times \tilde{T}$ closed equiv rel

$\rightsquigarrow$ $\mathrm{CHaus} \hookrightarrow \{\text{diamonds}/S\}$

$$T \longmapsto (X \longmapsto \mathrm{Cont}(|X|, T))$$

(e.g $[0,1]$ is a diamond)

No alg geometry, motivation for condensed math

This course: want diamonds related to alg geometry

$\Rightarrow$ Spatial diamonds

Def'n A diamond $Y = X/R$ is

1) spatial if it's qcqs ($\Rightarrow$ can choose X, R qcqs)

and $|Y|$ is spetral

and $|X| \to |Y|$ is spetral

- inv limit of finite top spaces
- $\simeq |\mathrm{Spec}\, A|$
- has a good behav of qc open subsets

preimage of qc open is qc open

2) locally spatial if it has an open cover (open subfunctors) by spatial $U \subset Y$

$\leadsto$ $|Y|$ is locally spatial

In practice, all diamonds are locally spatial (in FS)

Rek
- {locally spatial diamonds} has all (fibre) prods, all cofilt limit with qcqs transition maps
- spatial $\Leftrightarrow$ locally spatial + $|Y|$ qcqs

• pro-étale may not be open (e.g closed immersion)

Structure of a locally spatial diamond Y:

$|Y|$ $\quad \forall y \in |Y|$ $\exists$ localization $Y_y \subseteq Y$

$= \lim\limits_{U \ni y} \mathcal{U}$

$$Y_y = \mathrm{Spa}(C, C^+)/\underline{G}$$

C complete alg closed nonarch field

$m_{O_C} \subset C^+ \subset O_C$ valuation ring

G profinite gp acting continuously & faithfully on C.

(pro-p G wild part
prime-to-p G is classical tame part)

$$\{\text{Analytic adic spaces}/\mathbb{Z}_p\} \longrightarrow \{\text{diamonds}\}$$

$$X \longmapsto X^\diamond$$

Def'n/Prop'n $\forall$ analytic adic space $X/\mathbb{Z}_p$

$X^\diamond : \underset{\substack{\cap \\ \mathrm{Perf}}}{S} \longmapsto \left\{ S^\# / \mathbb{Z}_p \text{ untilt of } S + S^\# \to X \right\}$

is a $\underline{\text{locally spatial diamond}}$

and canonically $|X| \simeq |X^\diamond|$

$X_{\acute{e}t} \simeq X^\diamond_{\acute{e}t}$

$X \longmapsto X^\diamond$ remembers the top information of X

but forgets the structure map to $\mathrm{Spa}\,\mathbb{Z}_p$

If X perf'd then $X^\diamond \simeq X^\flat$

<u>Idea of the proof</u> If X perf'd then

tilting equivalence $\{S^\#, S^\# \to X\} \simeq \{S \to X^\flat\}$

$\rightsquigarrow$ $X^\diamond$ rep by X ...

In general, $\forall$ any X $\exists$

$$\tilde{X} \xrightarrow{} X$$

perfd $\qquad$ pro-étale surj

(locally $X = \mathrm{Spa}(A, A^+)$ $\mathcal{I}f$ $A/\mathbb{Q}_p$)

join $x^{\frac{1}{p^\infty}}$ $\forall$ all $x \in A^\times$

e.g. $x \in 1 + pA^+$

get perfd spaces

R

<u>Rek</u> (Kedlaya - Liu)

$$\left\{ \begin{array}{c} \text{seminormal rigid-analytic} \\ \text{spaces} / \mathbb{Q}_p \end{array} \right\} \xrightarrow{\text{fully faithful}} \left\{ \text{diamonds} / (\mathrm{Spa}\,\mathbb{Q}_p)^\diamond \right\}$$

is fully faithful

<u>Note</u> $(\mathrm{Spa}\,\mathbb{Q}_p)^\diamond(S) = \{ S^\# / \mathbb{Q}_p \text{ untilt of } S \}$

Back to relative FF curve

$\forall \quad S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}/\mathbb{F}_q$

$$\mathcal{Y}_{S,E} = \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+) \setminus \{[\varpi] = 0\}$$

$$\cup\!\!\mid$$

$$Y_{S,E} = \{\pi \neq 0\}$$

Q1: X, Y diamond

$|X \times Y| = ?$

$(X \times Y)_{ét} = ?$

Q2: How about $|X^\diamond|$ for X a non-analytic adic space

not a diamond

Is $X^\diamond$ diamond $\Leftrightarrow$ X analytic

Thm ("Diamond equation")

$$Y_{S,E}^\diamond \simeq S \times (\mathrm{Spa}\, E)^\diamond$$

abs prod no base

In other words

Given $T/\mathbb{F}_q$,

$$T^\# \longrightarrow Y_{S,E}$$

$$\Longleftrightarrow \quad T^\#/E \; + \text{ map } T \to S$$

pf: $\quad T^\# = \mathrm{Spa}(A, A^+)$

a map $T^\# \longrightarrow \mathcal{Y}_{S,E} \subseteq \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+)$

is the same as

$$W_{\mathcal{O}_E}(R^+) \longrightarrow A^+$$

s.t. $[\varpi]$, $\pi \longmapsto$ units of A

automatic as $T^\# / E$

Adjunction between $\underline{W_{\mathcal{O}_E}(\text{perf rings})}$ and $\underline{\text{tilting}}$

$$\Rightarrow \quad W_{\mathcal{O}_E}(R^+) \longrightarrow A^+$$

$$[\varpi] \longrightarrow \text{unit of } A$$

$$\Longleftrightarrow \quad R^+ \longmapsto (A^+)^\flat := \varprojlim_{x \mapsto x^p} A^+/\pi$$

$$\pi \longmapsto \text{unit of } A^\flat$$

$$\Longleftrightarrow \quad \underset{T^\# / E}{T} = \mathrm{Spa}(A^\flat, A^{\flat+}) \longrightarrow S = \mathrm{Spa}(R, R^+)$$

$\square$

Cor $|Y_{S,E}| \cong |Y^{\diamond}_{S,E}| \cong |S \times (\mathrm{Spa}\,E)^{\diamond}|$

$$\begin{array}{c} |S \times (\mathrm{Spa}\,E)^{\diamond}| \\ \downarrow \\ |S| \end{array}$$

Prop'n $\forall$ $S' \subset S$ open aff'd subset

$Y_{S',E} \hookrightarrow Y_{S,E}$ open immersion

with $|Y_{S',E}| = |Y_{S,E}| \times_{|S|} |S'|$ $\Box$

$\rightsquigarrow$ can glue $Y_{S,E}$ $\circlearrowleft \phi_S$ $\forall$ general $S / \mathbb{F}_q$

perfd space

s.t. $Y^{\diamond}_{S,E} \cong S \times (\mathrm{Spa}\,E)^{\diamond} / (\mathrm{Spa}\,E)^{\diamond}$

Def'n $X_{S,E} = Y_{S,E} / \phi_S^{\mathbb{Z}}$

$X^{\diamond}_{S,E} = S/\phi_S^{\mathbb{Z}} \times (\mathrm{Spa}\,E)^{\diamond}$

"relative Fargues–Fontaine curve"

Q & A:

Diamond by quasi-pro-étale relation?

"Eqs" the same

$$(\mathrm{Spa}\, E)^{\diamond} \qquad\qquad (\mathrm{Spa}\, E)^{\diamond} \times (\mathrm{Spa}\, E)^{\diamond} / \Sigma_2$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\mathrm{Div}^1_Y \qquad\qquad\qquad\qquad \mathrm{Div}^2_Y$$

Prop All Diamonds are v-sheaves

$\forall$ adic space $X/\mathbb{Z}_p$

$X^{\diamond}$ as v-sheaf:

$$S \longmapsto \{ S^{\#} \text{ untilt } + S^{\#} \to X \}$$

$$|X^\diamond| \to |X|$$

usually far from an iso

Ian Gleason

$$H^1(\mathcal{O}(-1)) = (\mathbb{A}^1)^\diamond / \underline{E}$$

$$S \mapsto H^1(X_{S,E}, \mathcal{O}(-1))$$

Q: $\mathrm{End}(\mathrm{Spd}\,\mathbb{Q}_p) = ?$

Q: Theory of alg group over $Y_{S,E}$?