11/13 Relative FF curve

Aim : Classify vector bundles on $X_{C,E}$

Last time - line bundles

- VB semi-stable slope 0

new proof : putting "geometric structure"

on $H^0(X_{C,E}, \mathcal{E})$ $\quad \mathcal{E} \in VB(X_{C,E})$

$H^1( \quad \ldots \quad )$ "Banach - Colmez Space"

Banach spaces Introduce the notion

Intermediate aim : Explain diamonds

+ relative FF curve for $S \in \mathrm{Perf}$

Ref : new version "etale cohomology of Diamonds"

Recall : perfectoid alg $/\mathbb{F}_p$ = perfect $\underbrace{\text{Tate alg}}_{\text{Huber's sence}}$ $/\mathbb{F}_p$ $\exists$ top nil unit $\varpi$

perfect Banach alg $/ \mathbb{F}_p((\varpi))$

perfd is always reduced
hence no de Rham theory!
it's very topological

perfd space $/ \mathbb{F}_p$ = adic space $/ \mathbb{F}_p$

covered by $\mathrm{Spa}(R, R^+)$

$R$ perfect Tate

$S = \mathrm{Spa}(R, R^+)/_{\mathbb{F}_q}$ aff perfd $\qquad E \supseteq \mathcal{O}_E \ni \pi$

$\mathbb{F}_q \rightsquigarrow$ residue field of $E$

can still define FF curve replacing $\mathcal{O}_C$ by $\underline{\underline{R^+}}$

Pick $\varpi$ in $R$ pseudounif

$$\mathrm{Spa}\, W_{\mathcal{O}_E}(R^+) \supseteq \underbrace{\mathcal{Y}_{(R,R^+),E}}_{\substack{\{[\varpi]\neq 0\} \\ \text{adic space}}} / \mathcal{O}_E \supseteq \underbrace{Y_{(R,R^+),E}}_{\substack{\{[\varpi]\neq 0,\ \pi\neq 0\} \\ \text{analytic adic space} \\ = \text{locally Tate alg}}} / E$$

radius func

$$\mathrm{rad}: \mathcal{Y}_{(R,R^+),E} \longrightarrow [0, \infty)$$

$$\phi_{R^+} \circlearrowright \underset{\cup\!|}{Y_{(R,R^+),E}} \longrightarrow (0,\infty) \circlearrowleft q$$

$$y \longmapsto \frac{\log|[\varpi](\tilde{y})|}{\log|\pi(\tilde{y})|}$$

max rk 1 generalization of $y$ valued in $\mathbb{R}$

$$\mathrm{rad} \circ \phi_{R^+} = q \cdot \mathrm{rad}$$

<u>Defn</u> $X_{(R,R^+),E} := Y_{(R,R^+),E} / \phi^{\mathbb{Z}}$

$\boxed{\text{Categorical quotient in loc ringed spaces}}$

relative FF curve

adic space $/E$

(not live on $S$ !)

<u>Exa</u> $E = \mathbb{F}_q((t))$

$$W_{\mathcal{O}_E}(R^+) = R^+[[t]]$$

$$\mathrm{Spa}(R,R^+) = \{\varpi \neq 0\} \subset \mathrm{Spa}(R^+,R^+)$$

$$\mathcal{Y}_{(R,R^+),E} = \mathrm{Spa}(R,R^+) \times_{\mathrm{Spa}\,\mathbb{F}_q} \mathrm{Spa}\,\mathbb{F}_q[[t]]$$

fiber prod ∃ with assumption!

$$\cup \qquad = \mathbb{D}_{\mathrm{Spa}(R,R^+)} \quad \text{open unit disc}$$

$$\cup$$

and aff × aff may not be aff!

non-qc

$$Y_{(R,R^+),E} = \mathbb{D}^*_{\mathrm{Spa}(R,R^+)} \quad \text{punctured open unit disc}$$

Note:

$$\begin{tikzcd}
Y_{(R,R^+)} \arrow[rr, hook] \arrow[dr] & & \mathcal{Y}_{(R,R^+)} \arrow[dl] \\
& \mathrm{Spa}(R,R^+) &
\end{tikzcd}$$

but not over $S$ after quotient by $\phi$

Claim: For general perfectoid

$$\mathcal{Y}_{S,E} \supseteq Y_{S,E}$$

can be glued from affinoid

$$Y_{S,E} \downarrow X_{S,E} = Y_{S,E}/\phi^{\mathbb{Z}}$$

easy for $E = \mathbb{F}_q((t))$:

$$\mathcal{Y}_{S,E} := S \times_{\operatorname{Spa}\mathbb{F}_q} \operatorname{Spa}\mathbb{F}_q[[t]]$$

Note $\phi \curvearrowright |S|$ by id $\rightsquigarrow |X_{S,E}| \longrightarrow |S|$

" Diamond Equation "

$$\mathcal{Y}_{S,E}^{\diamond} = S \times (\operatorname{Spa}\mathcal{O}_E)^{\diamond}$$

not a diamond but absolute diamond

$$\cup|$$

$$Y_{S,E}^{\diamond} = S \times (\operatorname{Spa}E)^{\diamond}$$

$$\downarrow$$

$$X_{S,E}^{\diamond} = Y_{S,E}^{\diamond} / \phi^{\mathbb{Z}}$$

Idea: Diamonds = quotient of perfd spaces (of char = p) by pro-étale relation

$$\{\text{Analytic adic space}/\mathbb{Z}_p\} \longrightarrow \text{Diamonds}$$

$$X \longmapsto X^\diamond$$

choose $\tilde{X} \longrightarrow X$ pro-étale cover

perfd space

$$X \text{``}=\text{''} \tilde{X}/R$$

$$X^\diamond := \tilde{X}^\flat/R$$

$$R \subseteq \tilde{X} \times \tilde{X}$$

perfd pro-étale equiv rel

(Pro) Étale maps of perfd space

Definition $f : Y \longrightarrow X$ between perfd space (possibly of mix char)

1) $f$ is finite étale if $\forall$

any open perfectoid affinoid $\mathcal{U} = \mathrm{Spa}(R, R^+) \subset X$

( equiv, for a cover )

( Open: open affinoid ? in perfd space, is perfectoid )

the preimage $V = f^{-1}(\mathcal{U})$

is aff perfd and

Rek. ( finite étale perfd is perfd ) (trivial in char p) tilting ⇒ mix char

(A) $\underline{S \text{ finite étale } R\text{-alg}}$

(B) $S^+ = \text{Int closure of } R^+$

$\Uparrow$ $\boxed{\text{(almost purity)}}$

$\Downarrow$

$\mathrm{Spa}(R, R^+)_{\text{fét}} \simeq (\mathrm{Spec}\, R)_{\text{fét}}$

2) $f$ is <u>étale</u>

( we don't have nilpotents in Perfd so no infinitesimal lifting crit )

just take a <u>cheaty</u> def

because we can talk about small discs

In analytic world (Huber) any étale between analytic adic space factors by open + fét

if locally on $X$

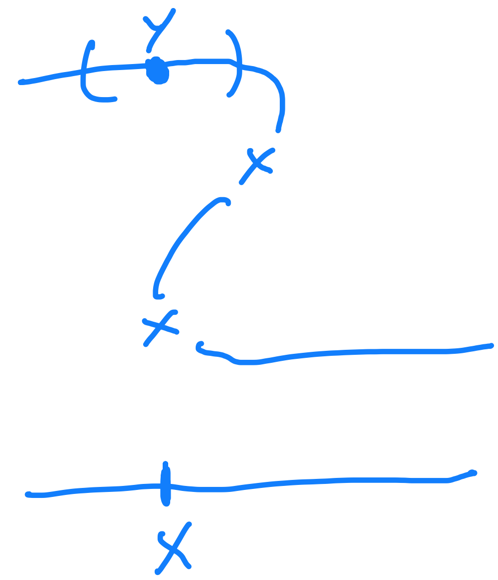
$$Y \xrightarrow[\text{open im}]{} Y' \xrightarrow[\text{finite étale}]{} \mathcal{U} \xhookrightarrow[\text{open im}]{} X$$

Warning: Nontrivial to see étale ∘ étale is étale (Thm of Huber)

• Not true in the world of schemes

ex

$Y$ :



$\downarrow$

$X$ :

3) $f$ is pro-étale if locally on $X$

affinoid pro-étale

$\exists$

$$Y = \mathrm{Spa}(S, S^+) = \lim_i \mathrm{Spa}(S_i, S_i^+)$$

$$\longrightarrow X = \mathrm{Spa}(R, R^+)$$

$f = \lim_i f_i$

s.t. all $f_i : \mathrm{Spa}(S_i, S_i^+) \longrightarrow \mathrm{Spa}(R, R^+)$

étale (similar notion replaced by finite étale)

$$S^+ = \left(\varinjlim S_i^+\right)^\wedge_{\varpi} \qquad S = S^+\left[\tfrac{1}{\varpi}\right]$$

History: "pro-étale topology of schemes"

Scholze – Bhatt

$\Uparrow$ motivates

pro-étale notion in "p-adic Hodge theory" Scholze

**Ex.**

$p \neq 2$

$$\begin{tikzcd} Y = \mathrm{Spa}(C\langle T^{\frac{1}{2p^\infty}}\rangle) \arrow[d, "\text{extract } \sqrt{T}"] \\ X = \mathrm{Spa}(C\langle T^{\frac{1}{p^\infty}}\rangle) \end{tikzcd}$$

**Claim** $\exists$ affinoid pro-étale cover $\tilde{X} \to X$

s.t.

$$\begin{tikzcd} \tilde{Y} \arrow[r] \arrow[d] & Y \arrow[d] \\ \tilde{X} \arrow[r] & X \end{tikzcd}$$

affinoid pro-étale (chill ram)

$U_n \subset X$ $:= \{ |T| \leq \frac{1}{p^n} \}$

$U_{n,n+1} = \{ \frac{1}{p^{n+1}} \leq |T| \leq \frac{1}{p^n} \}$

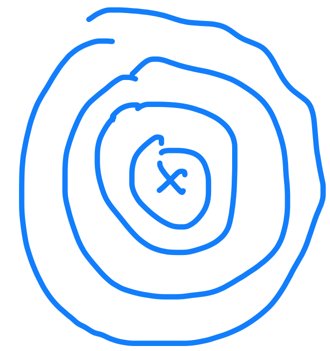
annulus

all affinoid perfd

For each $n$

Let

$$\begin{tikzcd} X_n = U_{0,1} \sqcup U_{1,2} \cdots\cdots \sqcup U_{n-1,n} \sqcup U_n \arrow[d, "\text{étale cover}"] \\ X \end{tikzcd}$$




$$\begin{tikzcd} X = \varprojlim X_n \arrow[r] \arrow[d] & X \\ \pi_0 Y = \mathbb{N} \cup \{\infty\} \end{tikzcd}$$

fiber over $n \in \mathbb{N}$ is $U_{n-1,n}$

$\infty$ is $\mathrm{Spa}\, C$ = origin of $X$

easy to see

$\tilde{Y} = Y \times_X \tilde{X} \longrightarrow Y$ is

<u>affinoid pro-étale</u>

It's in fact a good thing

not a local ring

just get the residue field

<u>Ex</u> Any Zariski closed immersion is affinoid pro-étale

$$X = \mathrm{Spa}(R, R^+) \supseteq V(f) = Z = \mathrm{Spa}(S, S^+)$$

$$S = R/\overline{(f^{1/p^\infty})} \supseteq S^+ = \text{int closure of } R^+$$

$$\mathcal{U}_n = \{|f| \leq \tfrac{1}{p^n}\} \subset X$$

rational open subset

$$\Rightarrow \bigcap_n \mathcal{U}_n = \varprojlim_n \mathcal{U}_n \subseteq X$$

$\| V(f)$

(perfectoid spaces are reduced, even uniform)

here $|f| = 0$ everywhere so $f = 0$

**Rek** In étale top for classical alg geometry

such "$\bigcap_n U_n$" will give strict henselization $\mathcal{O}^{sh}_{X,x}$

**Ex** pro-étale map has pro-finite fiber

**Thm.** $f: Y \to X$ of affinoid perfectoid spaces is pro-étale locally on $X$ affinoid pro-étale

iff $\forall$ geo (rk 1) pt

$$\mathrm{Spa}(C, \mathcal{O}_C) \to X$$

the fiber prod

$$Y \times_X \mathrm{Spa}(C, \mathcal{O}_C) \to \mathrm{Spa}(C, \mathcal{O}_C)$$

is affinoid pro-étale equiv, iso to

$$S \times \mathrm{Spa}(C, \mathcal{O}_C) \quad \text{for}$$

a profinite set $S = \varprojlim S_i$

Such $f$ is called quasi-pro-étale

Conclusion Can check it fiberwise (no flatness?)

**Def'n** $f: Y \to X$ $Y$ - cover meaning?

if $\forall$ any qc $U \subseteq X$

$\exists$ qc $V \subseteq Y$

s.t $|V| \to |U|$ surjective

**pro-étale cover** if v-cover + $f$ is quasi-pro-étale

**Thm** 1) $X \mapsto \mathcal{O}_X(X)$ are sheaves for

$\mapsto \mathcal{O}_X^+(X)$ v-topology

$\forall$ $X$ perfd 2) $\mathrm{Hom}(-, X)$ is a sheaf for v-topology

3) $X \mapsto VB(X)$ v-stack

4) $X$ affinoid perfd

then $H_v^i(X, \mathcal{O}_X) = 0$ $\forall i > 0$

$$H^i_v(X, \mathcal{O}_X^+) = 0 \qquad \forall\, i > 0$$

almost

killed by $\varpi^{1/p^n}$ $\forall\, n > 0$

## Sketch

prove that

$X \longmapsto \mathcal{O}_X^+(X)$ sheaf for étale top

$\Uparrow$ de Jong - van - der Put

$+ \quad H^i_{et}(X, \mathcal{O}_X^+) \underset{\text{almost}}{=} 0 \quad \forall\, i > 0$

classical $\Rightarrow$ open top

$+$ finite étale top

$\Uparrow$ almost purity

$\leadsto$ similar for $\mathcal{O}_X^+/\varpi$

only at this level commutes with lim

extends to aff pro-étale by filtered colimit

$\leadsto$ get $\mathcal{O}_X^+/\varpi$ has good properties as pro-étale sheaf

$\leadsto \quad \mathcal{O}_X^+ = \lim_n \mathcal{O}_X^+/\varpi^n \quad \checkmark$

$$\mathcal{O}_X = \mathcal{O}_X^+[\tfrac{1}{\varpi}] \quad \checkmark$$

not true at $\mathcal{O}_X^+$-level

get pro-étale desent

pro-étale locally, v-covers are faithfully flat on $\mathcal{O}_X^+/\varpi$-level

(so simple! you can split space to very simple shapes in pro-étale top)

then use faithfully flat descent $\square$

$\Rightarrow$ v-descent

Q&A:

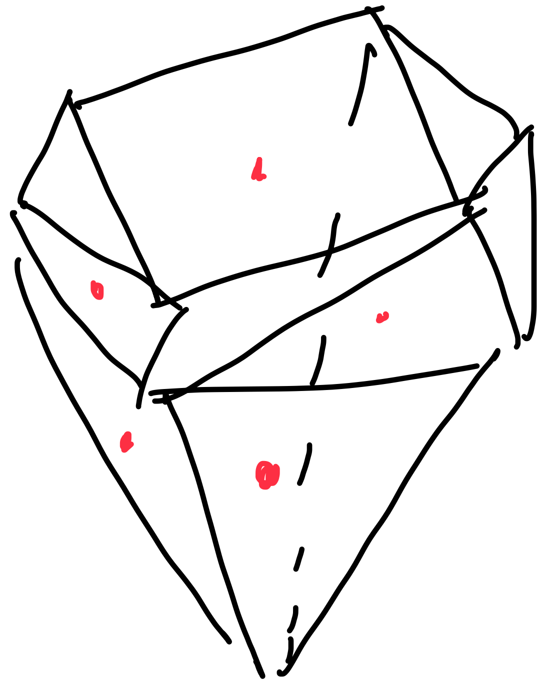
v-diamonds $\overset{\text{gives}}{\rightsquigarrow}$ all v-sheaves

topological issues for $\pi_1^{\text{pro-et}, v}$ not good

cohomology of $\mathcal{O}_X, \mathcal{O}_X^+$ are the same for v and pro-étale

v-descent for $\mathcal{O}_X^+$.

Next time



diamonds

$X \longrightarrow \pi_0(X)$

profinite

$\mathrm{Spa}(C, C^+)$

adic con comp

$\underline{\mathrm{Spa}(C, C^+)}$

$X = \tilde{X} / R$