02/08

Set up: $E$ nonarch local field

$G/E$ reductive

LLC:

\[ \begin{array}{c}
\{ \text{irr, smooth } G(E)\text{-rep} \} \\
\mapsto \\
\{ \text{L-parameter} \}
\end{array} \]

\[ \pi \mapsto \varphi_{\pi} \]

Usually, work with $\mathbb{C}$-coefficients

$\Rightarrow$ canonical $\sqrt{q} \in \mathbb{C}$

$L$-gp:

\[ L_G : G/E \to \text{dual gp } \hat{G} \big/ \hat{\Pi} \]

Defn:

\[ L_G = \hat{G} \times \mathbb{Q} \text{ alg gp} \big/ \hat{\Pi} \]
An $L$-parameter \( \overline{C} \) is a continuous map

\[
W_E \rightarrow L\mathcal{G}(\overline{C})
\]

there is an equiv. a continuous cocycle

\[
W_E \rightarrow \hat{\mathcal{G}}(\overline{C})
\]

Rek Continuity \( \Rightarrow \) factors over a discrete quotient \( W_E/I' \)

\( I' \subseteq I_E \) open finite index subgps

Deligne: It's better to keep track of monodromy operator
Take 2) An \( L \)-parameter \( / C \) is a pair \( (\varphi, N) \), where

\[ \varphi: W_E \rightarrow LG(C) \quad \text{cont gp} \]

\[ N \in \text{Lie} \hat{g} \otimes C \]

s.t \( \forall w \in W_E \quad \text{Ad}(\varphi(w))N = q^{L(w)}N \)

(or \( q^{-L(w)}N \)?)

(For \( G = GL_n \), these are so-called Weil-Deligne reps)

(Take 3, uncommon)

**Def’n** An \( L \)-parameter \( / C \) is a pair \( (\varphi, r) \), \( \varphi: W_E \rightarrow LG(C) \quad \text{cont gp} \)

\[ r: SL_2 \rightarrow \hat{g} / C \quad \text{alg rep} \]
sit $r, \varphi$ commute ($W_E \times SL_2 \rightarrow L(G)$)

Then

$$\varphi'(w) = \varphi(w) r \left( \begin{array}{cc} q^{(w)/2} & \{} \\
& q^{-(w)/2} \end{array} \right)$$

with $N = (Lir) \left( \begin{array}{cc} 0 & 1 \\
0 & 0 \end{array} \right)$ gives an

$L$-parameter in Take 2

Each Take 1, 2, 3

is a moduli of $L$-parameter

all distinct $L$-parameters in

Take 2, Take 3 are (up to $\hat{G}(CC)$-conj)

bijection

but scheme structure for 2, 3 are different
Reason: In Take 2, \( N \neq 0 \) degerates to \( N = 0 \)

In Take 3, \( SL_2 \) has "rigid" rep theory

Deligne's motivation: Fix \( C \cong \overline{Q}_L \)

(Take 2') \( L \)-parameter/\( \overline{Q}_L \) is a continuous gp homo \( \psi: W_E \to L \cdot \mathcal{G}(\overline{Q}_L) \)

Thm (Artin-Prasad-Deligne) Take 2 \( \iff \) Take 2'
Goal: Construct a moduli space of $L$-parameters, i.e., scheme locally of finite type

\[ Z^1(W_E, \hat{G}) / \mathbb{Z}_l \]

s.t. $A$-valued points

(A any $\mathbb{Z}_l$-alg)

are the continuous gp homos

\[ \mathcal{Y} : W_E \rightarrow L\hat{G}(A) \]

Dat-Helm-Kurinczuk-Moss, i.e., continuous 1-cycles

\[ W_E \rightarrow \hat{G}(A) \]

Obvious question: What topology on $A$?
Construction: Any $\mathbb{Z}_L$-mod $M$ can be endowed with the fift colimit topology

$$M = \lim_{\to} (M', \text{L-adic})$$

$$M' \subset M$$

$$f \cdot g \mid \mathbb{Z}_L$$

[In language of condensed math]

$$\underline{M} = M_{\text{disc}} \otimes_{\mathbb{Z}_L, \text{disc}} \mathbb{Z}_L$$

Refl. The moduli has no derived structure

Thin. There is a scheme $\mathbb{Z}^A(W_E, \hat{G})/\mathbb{Z}_L$ param L-parameters for $G$
It's a disjoint union of affine schemes of finite type over \( \mathbb{Z}_L \) that are flat, complete intersections, and of \( \dim \mathring{G} = \dim \hat{G} \).

Note: can divide by conjugation action of \( \hat{G} \) to get an Artin stack \( \text{"Loc Sys}\hat{\mathring{G}} \).

Rek. The natural ext to animated \( \mathbb{Z}_L\)-alg gives same moduli.

**Proof (Sketch)**: Any cont 1-cycle

\[ \gamma : W_E \to \hat{\mathring{G}}(A) \]

is trivial on an open subgp \( \mathfrak{p} \) of wild inertia

\[ \mathcal{Z}^1(W_E, \hat{\mathring{G}}) = \bigcup_{\mathfrak{p}} \mathcal{Z}^1(W_E/\mathfrak{p}, \hat{\mathring{G}}) \]
transition maps are open and closed

enough: All $\tilde{Z}^1(W_E/IP, \hat{\mathcal{G}})$ are affine, flat, complete int of $\dim = \dim \hat{\mathcal{G}}$

Trick: Pick $W \subset W_E/IP$ dense discrete subgroup of following form:

pick generators $\sigma \in W_E$ Frob

$\mathbb{Z} \subset I \subset \text{generator of tame inertia}$

Taking subgroup generated by $\sigma, \mathbb{Z}$, wild inertia generated by $\sigma$

$1 \rightarrow I \rightarrow W \rightarrow \mathbb{Z} \rightarrow 1$

gen by $\mathbb{Z}$

$1 \rightarrow (\text{finite } p\text{-gp}) \rightarrow I \rightarrow \mathbb{Z} [\frac{1}{p}] \rightarrow 1$
Claim \[ \mathbb{Z}^1(W_E/\mathcal{P}, \hat{\mathcal{G}}) \rightarrow \mathbb{Z}^1(W, \hat{\mathcal{G}}) \]
is an isomorphism.

Proof enough to show:

a cocycle \( \varphi_0 : W \rightarrow \hat{\mathcal{G}}(A) \) extends uniquely
to a cont. cocycle

\[ \varphi : W_E/\mathcal{P} \rightarrow \hat{\mathcal{G}}(A) \]

uniqueness: \( W \subseteq W_E/\mathcal{P} \) dense

existence: may enlarge \( E \), need to see

for any \( \ell \mathbb{Z}^{1/p^1} \times \mathcal{E} \mathbb{Z} \rightarrow \mathcal{G} \mathcal{L}_n(A) \)

\[ G \cdot \ell \mathcal{E} = \ell \mathcal{E} \]

the map \( \mathbb{Z}[\frac{1}{p}] \rightarrow \mathcal{G} \mathcal{L}_n(A) \)

\[ n \rightarrow \text{im}(\ell)^n \]
Extends continuously to \( \bigcup_{\ell \neq p} \mathbb{Z}_\ell \)

Note: \( \text{im}(\ell) \) conj. to \( \text{im}(\ell)^{\mathbb{Q}} \)

\( \Rightarrow \) all eigenvalues are roots of unity of order \( \text{prime} \times p \).

\( \Rightarrow \) some power is unipotent

But for unipotent matrices, all \( \mathbb{Z}_\ell \)-power are well-defined.

(\text{Claim}) \( \Rightarrow \mathbb{Z}^1(\text{WEIP, } \hat{\mathcal{A}}) \) affine scheme finite type complete intersection

\( \text{WEIP} \) has cohom \( \dim \leq 2 \)

to prove flat + correct \( \dim \),

enough to bound the \( \dim \) of special fibre.
There are finitely many unipotent conjugacy classes

\[ \dim \hat{\mathfrak{g}}_m \cap \mathfrak{z} + \dim \mathfrak{C}_m(\mathfrak{z}) = \dim \hat{\mathfrak{g}}. \]

A presentation of \( \mathcal{O}(Z^1(W_E(P, \hat{\mathfrak{g}})) \)

Fix discretization \( W \subset W_E | P \)

Then for any map \( F_n \rightarrow W \) from a free \( \mathfrak{g} \)-algebra \( F_n \)

get map

\[ Z^1(W_E(P, \hat{\mathfrak{g}})) = Z^1(W, \hat{\mathfrak{g}}) \]

\[ \rightarrow Z^1(F_n, \hat{\mathfrak{g}}) = \hat{\mathfrak{g}}^n \]
Prop \[ \varprojlim (\mathcal{O}((\hat{\mathcal{G}}^n))^\wedge) \overset{\sim}{\longrightarrow} \mathcal{O}(z^1(\mathcal{W}_E/\mathcal{P}, \hat{\mathcal{G}})) \]

(shifted \ outset \ \text{\textit{alim}} \text{\ (so agrees \ in \ mod/alg)}

Corollary

The map

\[ \varprojlim (\mathcal{O}((\hat{\mathcal{G}}^n))^\wedge) \overset{\sim}{\longrightarrow} \mathcal{O}(z^1(\mathcal{W}_E/\mathcal{P}, \hat{\mathcal{G}})) \]

is \ universal \ homeomorphism \ on \ spectra \ and \ an \ isom \ after \ inverting \ \mathbb{L}

(Use \ Haboush's \ thm \ on \ geom \ reductivity.

This \ will \ appear \ as \ "the \ algebra \ of \ excursion \ operators"
Thm: the map
\[
\varinjlim_{n, F_n \to W} \mathcal{O}(\hat{G}^n) \hat{G} \rightarrow \mathcal{O}(Z^A(W \vee P, \hat{G}))
\]
\(\hat{G}\)-action of simultaneous twisted conjugation

is an isomorphism if

\(\hat{G}\) der \(\Rightarrow\) \(Z(G)\) connected

and \(L\) is "not too small"

- all \(L\) type \(A\)
- all \(L = 2\) type \(2A_n, B_n, C_n, D_n, 2D_n\)
- all \(L = 2, 3\) type \(3P_4, 6D_4, E_6, E_7, F_4, G_2\)
- all \(L = 2, 3, 5\) type \(E_8\)
\( Q, \ (Pat) \)  \text{last then} \ Z(G) \text{ non-connected}

\( Q, \ (Zhiyin) \) Richardson's thy on closed orbits Semisimple

\( Q, \ (Yajie) \) deformation theory \text{Tate duality} \text{(Tate duality)}

\[ O(Z^1(W = L, \hat{G})) \text{ has a good} \]
\[ \Rightarrow H^i(\hat{G}, \mathcal{O}(Z^1)) = 0 \text{ } \forall i > 0 \]
\[ \text{G-fil} \]

\(\Rightarrow\) formalization of \( \hat{G} \)-invariant commutes with any base change

\( Q, \ (Le Bras) \) \( Z(G) \) conneted \text{ all in form from class}
arg. (le bras) colim in derived cat thm is still true D(Z)