\[ \text{Det} (\text{Bun}_G, \Lambda) \]

\( G/E \) reductive gp
\( \Lambda \) coefficient ring, \( n\Lambda = 0 \)

(recall \( \mathcal{O}_q(\text{Drinfeld}) \sim \mathbb{Z}_q \))

\( \text{char} = p > 0 \)
\( (n, p) = 1 \)

\( \Lambda \) has infinite semi-orthogonal decomposition into

\( \text{Det} (\text{Bun}_G, \Lambda) \sim \text{D}(\text{G}_b(E), \Lambda) \)

\( \S 1 \) compact object

Recall \( A \in C \) is compact iff

\( \text{Hom}_C (A, -) \) commutes with all direct sums

Fact \( C \) homotopy category of a stab \( \infty \)
category \mathcal{C} with all colimits, then

A \in \mathcal{C} compact \iff A \in \mathcal{C} \text{ satisfies } \text{Hom}_\mathcal{C}(A, -) \text{ commutes with all colimits} \iff \text{exact}

If \mathcal{C} is generated under colimits by its compact objects \mathcal{C}^w \subset \mathcal{C}, then

\text{Ind}(C^w) \cong \mathcal{C} \text{ equiv of \infty-cats}

\text{prop } D(C^w(E), N) \text{ is compactly generated}

compact generators are \text{span} \text{ of } \text{c-Ind}_K \text{ of } \text{G}_b(E)

\text{proof } \text{Hom}_\mathcal{C}(\text{c-Ind}_K \text{G}_b(E), N) \cong \text{Hom}_K(A, -) = (-)_K

Q: for which cpt \ K, (-)^K is compact.
\( K \text{ proj } \Rightarrow \text{ taking } K\text{-inv commutes with all direct sums} \)  
\((= K\text{-cohom})\)

(If \(A\) repr. by  
\[\ldots \to A_2 \to A_1 \to A_0 \to A_{-1} \to \ldots\]  
\(\Rightarrow A^K\) repr. by  
\[\ldots \to A_2^K \to A_1^K \to A_0^K \to A_{-1}^K \to \ldots\)  

So \(c\text{-Ind}_{K}^{G}(E)\) is compact \(\forall K\)

If \(A\) s.t. \(A^K = 0 \forall K\), then \(A = 0\)  
so generate \(\mathcal{O}\)

Thm \( Det(Buna, \Lambda)\) is cpt generated  
\(A \in Det(Buna, \Lambda)\) cpt iff \(\forall b \in B(\mathcal{G})\) \(\exists b^* A\) is compact and \(= 0\) \(\forall\) almost all \(b\)
First, exhibit compact generators

Fix $b \in B(A)$, $K \subseteq G_b(E)$ open proper.

Goal: Show that $\exists A^b_K \in \text{Det}(Bun_{\mathcal{A}}, \Lambda)$

Set $\text{RHom}(A^b_K, B) = (i^b \ast B)^K \quad \forall B \in \text{Det}(Bun_{\mathcal{A}}, \Lambda)$

if $\exists A^b_K$, it's cpt and these objs generates

to find $A^b_K$, use

$\pi_b: \tilde{M}_b \to \text{Bun}_{\mathcal{A}}$

$\tilde{M}_b$

$f_K: \left[ \tilde{M}_b / K \right] \to \text{Bun}_{\mathcal{A}}$

Claim: $A^b_K = Rf_K! Rf_K^! \Lambda$ works
pf: \[
\text{RHom}(A_k^b, B) = \text{RHom}(Rf_k! Rf_k^! \Lambda, B) \\
= \text{RHom}(\text{Rf}_k^! \Lambda, \text{Rf}_k^! B) \\
\text{f}_k \text{ULA} \\
= \text{RHom}(\text{Rf}_k^! \Lambda, f_k^* B \otimes \text{Rf}_k^! \Lambda) \\
\text{Rf}_k^! \Lambda \text{ inv} \\
= \text{RP} \left( \hat{M}_b/\mathbb{K}, f_k^* B \right) \\
\text{"flow"} \Rightarrow \hat{M}_b \text{ strict local} \\
= \text{RP} \left( \mathbb{C}^*/\mathbb{K}, f_k^* B \right) \left| _{\mathbb{C}^*/\mathbb{K}} \right) \\
= (i_b^* B)^K \text{.}
\]

For characterization of cpt objs, argue by induction on quasi-cpt open substacks

\[ U \subset \text{Bun}_G \]

Pick some \( b \in B(G) \) with

\[ i_b: \text{Bun}_G^b \rightarrow U \text{ closed} \text{ on } \text{Bun}_G \text{ no} \]
\[ \text{Bun}_G^b \text{ is closed, but on qc open it's ok} \]
j: \( V = U \setminus \text{Bun}_b^\mathbb{A} \to U \) 

we know the result for \( \text{Det}(V, \Lambda) \) 

enough: \( j^* \) preserves compact objects.

Indeed, then if \( A \text{ cpt} \) then \( j^* A \text{ cpt} \) 

\( j^* A \text{ cpt} \Rightarrow i_b^* A \text{ cpt} \forall b \neq b \) by induction

+ \( i^* A \) compact. So it's enough to show

Claim: \( j^* A_K^b \in \text{Det}(V, \Lambda) \) is cpt

Proof: \( f_k: [\tilde{M}_b / K] \to U \subseteq \text{Bun}_b^\mathbb{A} \)

\( f_k: [\tilde{M}_b^0 / K] \to V \)

\( \tilde{M}_b^0 = \tilde{M}_b \setminus * \) spatial diamond of finite dim try
\[ j^* A_K^b = Rf_{K!}^* Rf_{K}^* A \] by formula for \( A_K^b \).

By similar computation,

\[ R\text{Hom} ( j^* A_K^b, B) \cong R\mathcal{P} ([\tilde{M}_b^0 / K], f_{K}^* B) \]

\[ = R\mathcal{P} (\tilde{M}_b^0, \text{ pullback of } B)_{K} \]

But \( R\mathcal{P} (\tilde{M}_b^0, -) \) commutes with all direct

sums as \( \tilde{M}_b^0 \) is a spatial (qcqs!)

diamond (of finite dimension)

(need finite dim to make sure some convergence)

\[ \text{tricky} \]

\[ \text{§2. Bernstein-Zelevinsky duality} \]
A duality on cpt objs

\[ \forall A \in DCG_b(E), \Lambda)^w, \]
\[ \exists A' \in DCG_b(E), \Lambda)^w \]
\[ \text{such that } R\text{Hom} C_b(E), A', B) = (A \otimes B) C_b(E) \]

For \( A = \text{c-Ind}_K^{C_b(E)} \Lambda \)
get \( A' = \text{c-Ind}_K^{C_b(E)} \Lambda \)

In general
\[ A' = R\text{Hom}_{C_b(E)} (A, H(C_b(E))) \]

\[ (A')' \to A \]

is an isomorphism

Hecke alg of cpt supported locally constant functions on \( C_b(E) \)
proof: Yoneda: $A'$ unique if it exists

existence: enough to take $A = c \text{-} \text{Ind} \left( G_{\mathcal{E}} \right)_K$

$(A \otimes B) \cdot G_{\mathcal{E}} = B_K \cong B_K = B$, averaging

Thm: $\forall \text{ any } A \in \text{Det}(\text{Bun}_{G^L}, \Lambda)^w$,

$\exists! A' = \text{ID}_{B_{\mathcal{Z}}} (A) \in \text{Det}(\text{Bun}_{G^L}, \Lambda)$ s.t.

$\text{RHom} \left( \text{ID}_{B_{\mathcal{Z}}} (A), B \right) \cong \pi_{L_{\mathcal{L}}} (A \otimes \Lambda)$

Here, $\pi_{L_{\mathcal{L}}}$ is the left adjoint to $\text{Tr}^*$

$\text{RTr} \left( - \otimes \text{RTr}^! \Lambda \right)$
biduality map $\text{ID}_{B^2} (\text{ID}_{B^2}(A)) \to A$
is an isomorphism

Q: $\pi_G(A \otimes -$) for general $A$ not representable?

For $U = \text{Bun}^b_G$, $b$ basic, reduces to usual $BZ$-duality on $\text{Det} (\text{Bun}^b_G, \Lambda)^w = D (G_b(E), \Lambda)^w$

Q: How does $\text{ID}_{B^2}$ change the support?

proof. Check existence for a class of generators. Take $i^b: [\text{C-Ind}_{K}^{G_b(E)} \Lambda] \to \text{Bun}^b_G \to \text{Bun}_G$.
Claim: $\mathcal{D}_{B^Z}(i^b_! [c-Ind_{K} G_b(E)]^\Lambda) = A^b_K$

$\mathcal{RHom}(A^b_K, B) = (i^b_! B)^K$

$$= \mathcal{P}_{B^Z}(i^b_! [c-Ind_{K} G_b(E)]^\Lambda \otimes B)$$

up to shift

Need to check biduality i.e.

$\mathcal{D}_{P^Z}(A^b_{K}) \cong i^b_! [c-Ind_{K} G_b(E)]^\Lambda$

OK on $\text{Bun}_a^b$, need to check that

after pullback to complement, LHS = 0

$j: U \hookrightarrow \text{Bun}_a$ open subset

proper

($= \text{generalization of } b$)
to see: $\forall B \in \text{Det}(U, \Lambda)$

$R\text{Hom}(\text{Pr}^b, \text{R}j_\ast B)_{ji} \cong 0$

$\Pi_j(C \otimes \text{R}j_\ast B)$

$\forall$

$\text{R}T^c(\text{Pr}_b^b / K, \text{R}j_\ast B)_{c, j, b} = \text{R}\gamma_{\text{hom}} \text{ of } \text{R}\gamma^c(\text{Pr}_b^b / K, \text{R}j_\ast B)$

pull back of

with cpt supp

towards boundary of

$C\{\text{Pr}_b^b / K\}$, no supp

condition near

$E_{12}(\alpha / K) \in C\{\text{Pr}_b^b / K\}$

so it's zero by

"partial cpt supp vanishing"
Verdier duality \( \pi : \text{Bun}_a \to * \)

\[ A \mapsto R\text{Hom}(A, R\pi_1^! \Lambda) \]

is contravariant endofunctor on \( \text{Dét}(\text{Bun}_a, \Lambda) \)

Verdier duality

On \( \text{Dét}(\text{Bun}_b, \Lambda) \cong D(\text{C}_b(E), \Lambda) \) is just smooth duality (up to shift \( R\text{Hom}(\Lambda, \cdot)^! \))

Dualizing complex \( \cong \) Haar measures \( \uparrow \), \( \text{Haar measure on } H' \)

Dualizing complex \( \cong \) Haar measures \( \uparrow \), \( \text{Haar measure on } H' \)

Thm \( \forall \) any open imm \( j : U \hookrightarrow V \)

of open substacks of \( \text{Bun}_a \)

\( A \in \text{Dét}(U, \Lambda) \), we have
\[ R j_* \, R\text{Hom}(A, \mathbb{1}_{U}) \cong R\text{Hom}(j_! A, \mathbb{1}_{U}) \]  

(easy)

and \[ j! \, R\text{Hom}(A, \mathbb{1}_{U}) = R\text{Hom}(Rj_* A, \mathbb{1}_{U}) \]  

(doesn't matter, just twist)

\text{Cor} \quad A \in \text{Det}(\text{Bun}_G, \Lambda) \text{ is reflexive i.e.} \quad A \cong \text{ID}(\text{ID}(A)) \quad \iff \quad \forall b \in B(G), \ i_b^! A \in \text{Det}(\text{Bun}_G, \Lambda) \text{ is reflexive, i.e. } (i_b^! A)^K \in D(\Lambda) \]

(reflexive \quad \forall \text{ all } K \in \mathcal{C}_b(E) \quad \text{open p\_p}\)

\text{Thm} \quad \Rightarrow \quad i_b^! \text{ commutes with } \text{ID(\text{ID(\_))}}

proof of Thm. Can assume by induction \[ U = V \setminus \text{Bun}_G^b \]
(1) clear (6-functors)

(2) clear after $j^*$, so enough to show it's an isom after $R\text{Hom}(A_K \cdot -)$

As $R\text{Hom}(A_K \cdot B) = (i^* B)^K$ then $LUS = 0$

the right side of (2) just twist

$= R\text{Hom}(A_K \cdot Rj_* A, \Lambda)$

$= R\text{Hom} (A_K \bigotimes Rj_* A, \pi^* \Lambda)$

$= R\text{Hom}(\pi_b (A_K \bigotimes Rj_* A), \Lambda)$

enough: $\pi_b (A_K \bigotimes Rj_* A) = 0$

use the power of BZ duality

use $1D_{\mathfrak{g}} (A) = i^b_! [c\text{-Ind}_{K \cdot \Lambda}^b (\cdot)] Rj_* A = 0$
ULA sheaves

Bun_\mathcal{A}rtin \text{ v-stack} \Rightarrow \text{ notion of ULA sheaves}

\text{prop}'n \quad (\text{consequence of "dualizability" characterisation of being ULA })

A \in \text{Det} (\text{Bun}_\mathcal{A}, \Lambda) \text{ is ULA}

\text{iff} \quad p_1^* \text{RHom}(A, \Lambda) \otimes p_2^*A \cong \text{RHom}(p_1^*A, p_2^*A)

p_1, p_2: \text{Bun}_\mathcal{A} \times \text{Bun}_\mathcal{A} \to \text{Bun}_\mathcal{A}

\text{Thm} \quad A \in \text{Det} (\text{Bun}_\mathcal{A}, \Lambda) \text{ is ULA}

\text{iff} \quad \forall b \in B(G), K \subseteq G_b(E) \text{ open pro-p } (i_b^*A)^K \in \text{D}(\Lambda) \text{ perfect complex}
Lemma

Exterior $\otimes$ - prod

$\Box$ $\Delta_{\text{Det}(B_{\text{bn}}, \Lambda)} \otimes \Delta_{\text{Det}(B_{\text{bn}}, \Lambda)}$

$\Delta_{\text{Det}(B_{\text{bn}} \times B_{\text{bn}}, \Lambda)}$

is an equiv of $\infty$-cats

i.e $\forall A_1, A_2 \in \text{Det}(B_{\text{bn}}, \Lambda)$

also $A_1 \Box A_2 \in \text{Det}(B_{\text{bn}} \times B_{\text{bn}}, \Lambda)$

is compact

and $\forall B_1, B_2 \in \text{Det}(B_{\text{bn}}, \Lambda)$

RHom $(A_1 \Box A_2, B_1 \Box B_2)$

$\cong$ RHom $(A_1, B_1) \otimes$ RHom $(A_2, B_2)$

proof:

use cpt generators $\mathbb{A}_K$
proof of the theorem

need to check

\[ p_1^* \text{RHom} (A, \Lambda) \otimes p_2^* A \cong \text{RHom} (p_1^* A, p_2^* A) \]

apply \[ \text{RHom} (A_1 \otimes A_2, -) \] in \( A_i \) cpt

get:

\[ \text{RHom}(\pi_{\Lambda}(A_1 \otimes A), \Lambda) \otimes \text{RHom}(A_2, A) \]

\[ \text{RHom}(\pi_{\Lambda}(A_1 \otimes A), \text{RHom}(A_1, A)) \]

satisfied if \( \pi_{\Lambda}(A_1 \otimes A) \in \mathcal{D}(\Lambda) \)

\[ \iff \text{also} \text{ true, we test } A_1, A_2 \]

\[ A_1 = i_{b!} [c-\text{Ind}_{K}^{G_b(\mathbb{C})} \Lambda] \text{ to see this perfect} \]

translates to \( (i_{b!}^* A)^{K} \in \mathcal{D}(\Lambda) \)
Q: Drinfeld examples for $SL_2 / PGL_2$

$G = SL_2 :$

$\preceq \cdots \prec \odot O^2$

$O(-2) \oplus O(2) \oplus \cdots \oplus O(1)$

$E^x \quad E^x \quad Sl_2(E)$

Glueing $\mathfrak{g}$

Rep $SL_2$ and Rep $E^*$

Q: Zhu makes $(-1)$-val $-\mathfrak{g}$ opt

for all opt open $K$

Q: Jacquet-Langlands

Q: $G$ tori explicitly,
Q: (Drinfel'd) Use theory to compute $\text{Rep}$. 

Q: $T_v$ Hecke operator change the support.

(supercuspidal rep) only in ss loc.

Q: $\text{ID}_B^\tau$ for non-compact whatever $A$ is

$\mathcal{H}(A \otimes B)$ is cpt