$E \cong 0_E, \pi, \mathbb{F}_q$ as usual

**Thm** $S$ perfectoid space / $\mathbb{F}_q$

$Z \longrightarrow X_S \equiv X_{S,E}$ smooth map of (sous-perfectoid) adic spaces

$s \in Z$ quasi-projective (Zariski closed embedding $Z \hookrightarrow U \subset \mathbb{A}^n_{X_S}$)

Letting $M_Z \leq M_Z = \{\text{sections of } Z \to X_S \}$

be the open subspace $T / S \mapsto \{ X_T \to X_S \}$

where $s^*T_{Z/X_S}$ has everywhere $> 0$ HN slope
then the map $\mathcal{M}^\text{smooth}_Z \to S$ is $l$-cohom. smooth $\forall l \neq p$.

(Recall: $\mathcal{M}_Z \to S$ is repr. in loc spat. diamond, locally finite dim tag)

$\Rightarrow$ same for $\mathcal{M}^\text{smooth}_Z$

Strategy: 1) formal smoothness of $\mathcal{M}^\text{smooth}_Z$

2) formal smooth + "geo finite dim"

$\Rightarrow \{ F_i \text{ f-ULA} \} \Leftrightarrow f$ cohom. smooth

3) $Rf^! F_i$ invertible

1) "formal smoothness" $T_0 \to CT$

Zar closed im of aff'ld perf'd spcws

(important, can't assume just local diamonds)
$\Rightarrow \exists T' \rightarrow T$ étale containing $T_0$ in image

$s.t. \quad T_0 \times_T T' \rightarrow M^\text{smooth}_2$

pf: tricky, explicite BC spaces argument

2) Prop $\exists$ a $\text{ahom. smooth + formally smooth}$ subject to map $T_0 \rightarrow M^\text{smooth}_2$ locally finite étale compatifiable

st $T_0$ is a perf'd space st $T_0$ locally admit a Zariski closed embedding into finite dim perf'd balls

proof: $M^2_2 \subset M^{1p^n}_S$ locally Zar closed (in suitable sense)

$\text{c pullback of surj is surj }$

only need do it for $M^{1p^n}_S$
\[ M^{p^n}_S \subseteq \bigcup_{d>0} \text{ open } d > \varepsilon \] 

\[ \Rightarrow \text{ enough for } BC(O(d)^{n+1})_S | E^{x} \]

this can be done explicitly? 

Cor 
\[ F_V \text{ is ULA for } M^{sm}_Z \rightarrow S \]

prof: enough for \[ T_0 \rightarrow S \text{ (check locally)} \]

\[ \text{coh, smooth} \]

Lem \[ T_0 \rightarrow S \text{ map of aff'd perf'd space} \]

\[ s.t. \ (1) \ T_0 \rightarrow S \text{ formally smooth} \]

\[ 2) \ T_0 \rightarrow \text{LB}_S \text{ Zariski closed in a finite dim perf'd ball} \]

Then \[ F_V \text{ is ULA for } T_0 \rightarrow S \]
prof: \[ T_0 \times_T T' \longrightarrow T_0 \xrightarrow{\text{étale}} T = \operatorname{IB}_S^n \rightarrow S \]

\[ \Rightarrow \]

Shrinking \( T' \), can even find a retraction

\[ T' \rightarrow T_0 \times_T T' \]

\[ \Rightarrow \]

etale \[ T_0 \times_T T' \]

Retract of \[ T_0 \]

ULA \[ \leftarrow \]

ULA \[ \xrightarrow{\text{étale}} \]

ULA is étale local

( even abom. smooth )

being ULA is stable under retracts

follows either by defin

or from categorical characterisation
3) \( Rf^* \left( \mathcal{F}_\ell \right) \) is invertible is locally isomorphic to \( \mathcal{F}_\ell \mid_{\mathcal{X}} \).

Fact: If \( A \) is a \( \text{f-ULA} \) for \( f : X \to S \), then \( 1D \times S (A) \) is again \( \text{f-ULA} \) and its formalism commutes with any \( s' \to S \) base change.

Being invertible can be checked \( \nu \)-locally, so can be checked after pullback along \( \nu \)-cover.

\( \Rightarrow \) Passing to universal section of

\[ M^\text{smooth}_Z \to S \]

enough to prove that for a section

\[ s : S \to M^\text{smooth}_Z \]

\( \cong (s : X_S \to Z) \)
the pullback \( s^* Rf^! IF_\ell \subset \text{Det}(S, IF_\ell) \) is invertible.

Now we use deformation to the normal cone:

\[
\begin{array}{ccc}
S & \rightarrow & \sim \rightarrow \sim \rightarrow \sim \rightarrow Z \\
\downarrow & & \downarrow \text{smooth} & \downarrow & \downarrow \\
X_S & \rightarrow & X_S \times \mathbb{A}^1 \\
\end{array}
\]

s.t. \( \sim \times \mathbb{A}^1 \{1\} = Z \)

\( \sim \times \mathbb{A}^1 \{0\} \) = normal cone of \( s \) in \( Z \)

\( = \text{geometric vector bundle over to } s^* T_Z|_{X_S} \)

\[
\begin{array}{ccc}
E \times S & \rightarrow & \mathbb{A}^m \\
\cap & \rightarrow & \cap \\
X(S \times E) \rightarrow X_S \times \mathbb{A}^1 \\
\end{array}
\]
Get $\bar{\mathcal{Z}}' \to X \times S \times \mathbb{E}$ fibre over $S \times \{1\}$ is $\mathbb{Z}$ over $S \times \{0\}$ is $s^*T_2|_{X \times S}$

(0 is still in closure of $\mathbb{E}$ under action of $\mathbb{E}_x$)

$\tilde{\mathcal{Z}}'$ still quasi-proj, all previous results apply

$\tilde{f}: M_{\tilde{\mathcal{Z}}'} \to S \times \mathbb{E}$

and $R\tilde{f}^!\mathcal{F}_i$ is $\tilde{f}$-ULA

$R\tilde{f}^!\mathcal{F}_i |_{S \times \{1\}} = Rf^!\mathcal{F}_i$

$+ R\tilde{f}^!\mathcal{F}_i |_{S \times \{0\}}$ = dualizing complex is invertible for $BC(s^*T_2|_{X \times S})$
Q: explicit

\[ S^* Rf' \mathcal{F} \text{ is invertible} \]

deformation to the normal cone

(Similar argument by Clausen to show dualizing sphere of p-adic lie gp agrees and \ldots \text{ of its lie alg})

Application to \( \text{Det}(\text{Bun}_A, \mathcal{L}) \)

Recall charts for \( \text{Bun}_A \):

Def'n: Let \( M \) be the moduli space of \( \mathbb{Q} \)-filtered \( G \)-bundles, i.e.

\[
\text{exact } \otimes - \text{ functor } \text{Rep}_E G \xrightarrow{\rho} (\mathbb{Q} \text{-Fil Bun}_{A^s})
\] (increasing filtration)
$\text{st \: A \: all \: } V \in \text{Rep}_E \text{ G}$

$p(V)^\lambda := \bigcup_{\lambda' \leq \lambda} \bigcup_{V \text{ semi-stable of slope } \lambda'} p(V) \leq \lambda$

"opposite HN filtrations."

Then $\bigsqcup_{b \in B(G)} M_b = M \to \text{Bun}_G$

pass to associated graded bundle

$\bigsqcup_{b \in B(G)} \left[ * \big/ G_b(E) \right]$

Thm $M \to \text{Bun}_G$ is a hom, smooth

Example: $\text{GL}_2 \quad b = O \oplus O(1)$
Then \( M_b \) classify extensions

\[
0 \rightarrow L \rightarrow E \rightarrow L' \rightarrow 0
\]

\[
\deg L = 0 \quad \deg L' = 1
\]

\[ G = \text{GL}_n : \text{similar successive extensions} \]

Thm is a consequence of Jacobian criterion

(Take any \( S \rightarrow \text{Bun}_A \) \( \cong E / X_S \)

\( Z = \) moduli space of \( \mathbb{Q} \)-fil on \( E \)

Then \( M \in M_Z \) actually lie in

\[ M^\text{smooth} \] by condition on shapes.
Now fix be $\text{B}(\mathcal{G})$, consider

$$
\pi_b: \quad \mathcal{M}_b \longrightarrow \text{Bun}\mathcal{G}
$$

"chart for $\text{Bun}\mathcal{G}$ near $\text{Bun}_b^G$".

Structure of $\mathcal{M}_b$:

- \[ \mathcal{M}_b = [\tilde{\mathcal{M}}_b / G_b(E)] \]

In $\tilde{\mathcal{M}}_b$, graded bundle is trivialized

\[ \mathcal{E} \quad \text{e.g.} \quad \{ 0 \rightarrow 0 \rightarrow \mathcal{E} \rightarrow \mathcal{O}(1) \rightarrow 0 \} \]

"base point" \( * \in \tilde{\mathcal{M}}_b \) corresponding to split extension
\[ [\ast / G_b(E)] \subseteq M_b \]

\[ \xymatrix{ \ast \\ \text{Bun}_b^b \\ \text{Bun}_A \ar[u]_{\text{Bun}_b^b} } \]

(i.e. if \( E \in \text{Bun}_A^b \) then HN fil of \( E \) gives splitting of given \( Q \)-fil)

\[ \tilde{M}_b \rightarrow \ast \text{ rep in \ loc \, spot \ diamonds, \ cohom, smooth} \]

successive ext of negative BC spaces of \( \text{dim} = \langle 2p, V_b \rangle \)
\[ \{ 0 \to O \to E \to O(\frac{1}{2}) \to 0 \} \]

\[ = \text{BC}(O(-1)) [2] \]

- \( \tilde{M}_b \backslash \ast = \tilde{M}_b^0 \) is a spatial diamond

\( \Rightarrow \) qcqs! , but not qcqs!^2

In \( \text{GL}_2 \)-example, it's

\[
\frac{(\text{Spa} \, \overline{F_q((t^{1/2})}))/\overline{\text{SL}_2(D)}}{\text{aff'd perfectoid profinite}} \]

\( \text{alg} \)

(On \( \tilde{M}_b^0 \), \( E = O(\frac{1}{2}) \) Picking such iso

\[ \text{this is} \]

\[ \text{BC}(O(\frac{1}{2})) \backslash \{ 0 \} = \text{Spa} \, \overline{F_q((t^{1/2})}. \]

\( t \), so \( t \) can't go to zero
Warning. In \( GL_2 \)-example

modulo some \( \hat{M}_b \leq \hat{M}_b = \hat{M}_b^0 \cup \mathcal{B} \)

\( \text{gp action} \)

\( \overline{\text{Spa}(\mathbb{F}_q((t\frac{1}{T^m})))} \)

formal punctured open disc

after base change to \( \mathbb{C} \):

- \( \hat{M}_b \) "strictly local": as strict localization at \( b \)

\( \text{this point } * \) sits

\( \text{near } |t| = 1 \)

not near origin \( |t| = 0 \)!
If any \( A \in D_{et}(\tilde{\mathcal{M}}_b, \Lambda) \), the restriction
\[
R^\_\ast (\tilde{\mathcal{M}}_b, A) \to R^\_\ast (*, A)
\]
is an isomorphism.

**Sketch** the cone of this map is
\[
R^\_\ast_{c.s.} (\tilde{\mathcal{M}}^0_b, A)
\]
compact supported towards *
no supported condition towards boundary of \( \tilde{\mathcal{M}}_b \)

Special case of: e.g.

Let \( X = (\tilde{\mathcal{M}}^0_b) \) spatial diamonds dim \( \leq \omega \)
\[
\begin{array}{c}
X \to * \\
partially proper
\end{array}
\]
\( \xrightarrow{\text{`one-pt' compactification}} \)

i.e. \( X(R, R^+) = X(R, R^0) \)
Then for any $C/\mathbb{F}_q$

$X_C$ has "two ends"? precise examples: $X = \text{Spa } (R, R^+) \text{ aff'd perf'd}$

$C = \text{Spa } (\mathbb{F}_q ((t)))$

$X_C \rightarrow X \times \text{Spa } (\mathbb{F}_q ((t)))$

\[\text{profinite} \rightarrow \text{punctured open unit} \rightarrow \text{disc } / X\]

one boundary = origin
another boundary = "boundary of" unit disc

$\Rightarrow$ can define "partial compact supported cohomology"
\[ R_{\theta-c}^p (X_c, A) \]

**Thm**

\[ R_{\theta-c}^p (X_c, A) = 0 \]

**Sketch:**

Reduce to \( X = \text{Spa} \bar{F}_q ((t^{1/p^m})) \)

(Use proper base change + "correspondence")

\[ + A = \Lambda + \text{compute} \]

(Not many \( A \))

**Picture:**

Say \( M \) topological manifold

*free action* \( \mathbb{R} \circlearrowright M \) "flow"

\[ \text{s.t. } \overline{M} = M / \mathbb{R} \text{ is compact} \]
two boundaries  
"source of flow"

collapse to "end of flow"

+ for all $A \in D(M/\mathbb{R}, \mathbb{Z})$,  

$$R^{\partial}_{\sigma-c}(M, A) = 0$$

pf: Flow contracts everything  

"later for hyperbolic localization"

How is this analogous?

Roughly:  

$$C = \overline{F_q ((t^R))} \supseteq_{\mathbb{R}, 0} \text{exp}_{\mathbb{R}} \text{rescaling}$$
$X_C \subseteq \mathbb{R}$

$X_C \subseteq M$

$X_C/\mathbb{R} \rightarrow \mathbb{R}$

Prop: "ends"

Cor. $X_C$ as above satisfies "odd-dim' Poincaré duality"

$RP_C(X) \twoheadrightarrow RP_{\vartheta-c_1}(X) \otimes RP_{\vartheta-c_2}(X)$

$\Rightarrow RP_C(X) \simeq RP_C(X)$
Q: $T_0 \rightarrow M_2$ ?

shall all work in practice

If $X$ Zariski closed in finite-dim perfectoid ball $/C$

is $\dim X = \dim \text{tng} X$ ?