01/08 - Geometric Satake
- Jacobian Criterion

Set up $\quad S$ perfectoid space of char $p>0$

X_S FF curve

$Z \downarrow X_S$ smooth adic space

locally étale over a ball $/ X_S$

$\rightsquigarrow$ space of sections

Definition $\quad \mathcal{M}_Z : \{\text{perf'd spaces } T/S\} \to \text{Sets}$

$$T \longmapsto \left\{ \begin{tikzcd} & Z \arrow[d] \\ X_T \arrow[r] \arrow[ur, dashed] & X_S \end{tikzcd} \right\}$$

Example $\quad$ 1) If Z_0/E smooth

can take $\quad Z = Z_0 \times X_S$

then $\mathcal{M}_Z(T) = \mathrm{Map}(X_T, Z_0)$

"Gromov-Witten like space"

2) If $Z = \mathcal{E}$ geometric vector bund$/X_S$

then $\mathcal{M}_Z = BC(\mathcal{E})$

so $\mathcal{M}_Z$ in general is a "non-linear Banach-Colmez space"

3) $\mathcal{E}$ G-torsor on X_S

$P \subseteq G$ parabolic, then

$$\begin{array}{c} Z = \mathcal{E}/P \\ \downarrow \\ X_S \end{array}$$

geometric fibres $\simeq G/P$

(paritial) flag varieties

$\mathcal{M}_Z(T) =$ reduction of $\mathcal{E}|_{X_T}$ to $P \subseteq G$

this case will be used for the chart $\pi_b : M_b \to \mathrm{Bun}_G$

want to understand geometry of M_Z

<u>prop</u> If Z/X_S is quasi-projective

($\exists$ $Z \xrightarrow[\text{Zar closed}]{} U \underset{\text{open}}{\subseteq} \mathbb{P}^n_{X_S}$)

then M_Z is repr. in locally spatial diamonds $M_Z \to S$ compatifiable of locally finite dim.ty

<u>Conj</u> True for all Z/X_S <u>smooth</u> ?

<u>Sketch</u>. reduce to $Z = \mathbb{P}^n_{X_S}$

This can be explicit:

$$\mathcal{M}_{\mathbb{P}^n}(T) = \mathrm{Map}(X_T, \mathbb{P}^n)$$

$$= \{ (\mathcal{L}, s_0, \dots, s_n) \mid \mathcal{L} \text{ line bundle on } X_T,\ s_0, \dots, s_n \in H^0(X_T, \mathcal{L}) \text{ generating } \mathcal{L} \} / \cong$$

$$= \bigsqcup_{d \geq 0} \mathcal{M}_{\mathbb{P}^n, \deg = d}$$

$$\uparrow \quad \underline{E}^\times\text{-torsor : para iso } \mathcal{L} \cong \mathcal{O}(d)$$

$$\widetilde{\mathcal{M}}_{\mathbb{P}^n, \deg = d} \underset{\text{open}}{\subseteq} BC(\mathcal{O}(d))^{n+1}$$

$$\Rightarrow \mathcal{M}_{\mathbb{P}^n} = \bigsqcup_{d \geq 0} \underbrace{(\text{open subset of } BC(\mathcal{O}(d))^{n+1}) / \underline{E}^\times}_{\text{locally spatial diamonds of finite dim}}$$

not of finite dim only locally

Remark: See that $\mathcal{M}_{lp^r}$ is "almost" linear, This is a phemomenon for $G = GL_n$: spaces in 3) are "essentially" linear

But not for other gps! For classical gps, get "essentially quadritic" like $\{(x,y,z) \mid x,y,z \in H^0(\mathcal{O}(1)),\ x^2+y^2+z^2=0 \in H^0(\mathcal{O}(2))\}$

So can prove cohomological smooth of GL_n by hand

Goal Find large open subset

$$\mathcal{M}_Z^{sm} \longrightarrow \mathcal{M}_Z \quad \text{s.t}$$

$\mathcal{M}_Z^{sm} \longrightarrow S$ is cohomological smooth.

"Jacobian criterion": If $s : X_T \longrightarrow Z / X_S$ section get $\underbrace{s^* T_{Z/X_S}}$ $\in VB(X_T)$

tangent bundle VB on Z

Classically, deformation of s

$$\cong H^0(X_T, s^* T_{Z/X_S})$$

(global deform use global section)

obstructions

$$\cong H^1(X_T, s^* T_{Z/X_S})$$

<u>Idea</u> If $H^1(X_T, s^* T_{Z/X_S})$ vanishes, then s shall be a smooth pt

<u>Definition</u> $\mathcal{M}_Z^{sm} \subseteq \mathcal{M}_Z$ is the open sub functor of all s

s.t. $s^* T_{Z/X_S}$ has everywhere only

positive >0 HN slopes.

(not $=0$)

as $BC(\mathcal{O})$ not

coh smooth !

infinite profinite set

are not

Thm Jacobian criterion $\mathcal{M}_Z^{sm} \xrightarrow{f} S$ is cohomologically

smooth, and

$$(Rf^! \Lambda)_s \cong (\underbrace{R(f_s^{lin})^! \Lambda}_{\cong \Lambda(d)[2d]})_0$$

$$f_s^{lin} : BC(s^* T_{Z/X_S}) \longrightarrow \mathrm{Spa}\, C$$

$$\begin{array}{c} \psi \\ 0 \end{array}$$

Idea $\begin{array}{c} \mathcal{M}_Z^{sm} \\ \downarrow \\ s \end{array}$ $\underset{\text{inf loc near } s}{\sim}$ $\begin{array}{c} BC(s^* T_{Z/X_S}) \\ \psi \\ 0 \end{array}$

Application

Recall

$\mathcal{M}^{\diamond}_{LT,\infty} \cong \mathcal{M}^{\diamond}_{Pr,\infty}$ space of maps

$$\mathcal{O}^n_{X_S} \hookrightarrow \mathcal{O}_{X_S}\left(\frac{1}{n}\right)$$

s.t cokernel supported at ∞

i.e the given untilt

this is given by some $\mathcal{M}_{\mathbb{Z}}$

Ivanov – Weinstein

Jacobian criterion $\Rightarrow$ $\mathcal{M}_{LT,\infty} \setminus$ points with extra endomorphism is cohomological smooth

connected component

e.g $n=2$: complement of CM periods is cohomological smooth

In particular, cohomology of (qc open subset without CM pt) stabilizes in the tower

$$\begin{tikzcd} \vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] \\ \mathcal{M}_{LT,2} & \supseteq \mathcal{U}_2 \arrow[d] \\ \mathcal{M}_{LT,1} \arrow[u, leftarrow] \arrow[d] & \supseteq \mathcal{U}_1 \\ \mathcal{M}_{LT,0} & \end{tikzcd}$$

without CM pts

qc open

$$H^j(\mathcal{U}_1, \overline{\mathbb{Q}}_\ell) \hookrightarrow H^j(\mathcal{U}_2, \overline{\mathbb{Q}}_\ell) \hookrightarrow \cdots$$

transition map are eventually isom!

first "empirically" observed by Weinstein

How to prove Jacobian criterion ?

Naive idea : try to find direct geometric relation to $BC(s^* T_{Z/X\times S})$

this seems very hard

actual method has 3 steps

1) Definition of "formal smoothness" for maps of diamonds.

2) Definition of "universal local acyclicity"

ULA

$$\begin{array}{c} X \\ \downarrow f \\ S \end{array} \qquad A \in D_{et}(X, \Lambda)$$

Q: relation between formal smooth cohomological smooth

A "f-ULA":

some kinds of "flatness"

- f is cohomological smooth $\iff \begin{cases} \Lambda \text{ f-ULA} \\ Rf^! \Lambda \text{ is invertible} \end{cases}$

- formal smoothness + "geometric finite dim"

$\Rightarrow \exists$ f-ULA

3) deformation to the normal cone

Q: explicitly?

$\mathcal{M}_Z^{sm}$ $\overset{\text{degeneration}}{\dashrightarrow}$ $BC(s^* T_{Z/X_S})$

$\downarrow$ ... $\downarrow$

S ... 0

+ dualizing complex must be constant

needs an argument

uses ULAness

Formal smoothness:

idea: replace infinitesimal neighborhoods by small actual neighborhoods

Thm If
$$\begin{tikzcd} S_0 \arrow[r] \arrow[d, phantom, "\cap"] & X \arrow[d] \\ S \arrow[r] & Y \end{tikzcd}$$
Zar closed

smooth diagrams of adic spaces

~~S_0, S affinoid Y, X affinoid~~

$\exists\ U \subset S$ open containing S_0

lift
$$\begin{tikzcd} S_0 \arrow[r] \arrow[d, phantom, "\cap"] & Y \arrow[d] \\ U \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

Key $\varinjlim_{U \supset S_0} \mathcal{O}(U)$ henselian along I

$= \ker(\cdots \to \mathcal{O}(S_0))$

(classical analog : analytic funcs hensel..)

Definition Let $f : Y \longrightarrow X$ map of small v-stacks

Then f is formal smooth if fs

all Zar closed im of aff'd perf'd space $S_0 \hookrightarrow S$

$$\begin{tikzcd} S_0 \arrow[r] \arrow[d, phantom, "\cap" rotate=0] & Y \arrow[d, "f"] \\ S \arrow[r] & X \end{tikzcd}$$

$\exists$ étale map $S' \to S$ image containing S_0

and a lift

$$\begin{tikzcd} S_0 \times_S S' \arrow[r] \arrow[d, phantom, "\cap"] & Y \arrow[d, "f"] \\ S' \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

<u>Claim</u> This is related to

"absolute neighborhood retracts" (ANR)

Y compact Hausdorff is ANR (e.g. $\prod_I [0,1]$)

if $\forall$ any closed im $Y \hookrightarrow Z$

$\exists$ $U \subseteq Z$ open containing and a retraction $U \to Y$

Here, assume Y aff'd perf'd space

$$\begin{array}{c} Y \\ \downarrow \\ X = \mathrm{Spa}\, C \end{array}$$

Can embedd

$$\begin{array}{l} Y \\ \,\,\shortparallel \\ \,\,\,S_0 \hookrightarrow S = \mathbb{B}^I_C \end{array}$$

$$Y = \mathrm{Spa}(R, R^+) \subseteq \mathrm{Spa}(C\langle X_i\rangle, \mathcal{O}_C\langle X_i\rangle)$$

by choosing $\mathcal{O}_C\langle X_i \mid i \in I\rangle \twoheadrightarrow R^+$

S_0

$$\begin{tikzcd} S_0 \arrow[r, equal] \arrow[d, phantom, "\cap"] & Y \arrow[d] \\ S \arrow[r] \arrow[ur, dashed] & \mathrm{Spa}\, C \end{tikzcd}$$

ess. condition says $\exists\ U \subseteq S$ containing S_0

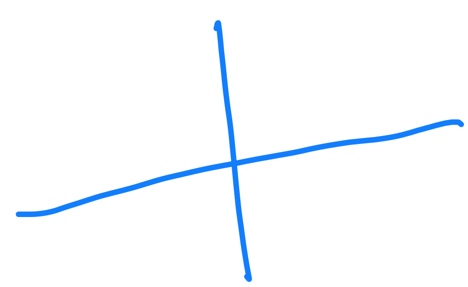
retraction $S \longrightarrow S_0 = Y$

Y formal smooth $\Longleftrightarrow$ retract of a space étale over a possibly ∞-dim ball

+ "geom. fin-dim'l"

Question: Assume Y aff'd perf'd space$/\mathbb{C}$ that is a retract of a space étale over a finite-dim ball. Is Y cohomological smooth?

↯ analogues fails for compt Hausdorff spaces:

 is ANR

But analogue is true for schemes

(implies finite pres + formal smoothness $\Rightarrow$ smooth)

$\boxed{\text{Thm}}$ $\mathcal{M}_Z^{sm} \longrightarrow S$ is formal smooth

Sketch estimate (only one in the paper)

$$\begin{tikzcd} S_0 \arrow[r] \arrow[d, phantom, "\cap"] & \mathcal{M}_Z^{sm} \arrow[d, "f"] \\ S \arrow[r, equal] & S \end{tikzcd}$$

Zar closed in of $\wedge$ affinoid perfd spaces

(which the test S in $S_0 \subset S$ agree with our S by base change to S)

(we can't reduce to S tot disconnected because we want finite dim)

i.e.

$$\begin{tikzcd} X_{S_0} \arrow[r, "S_0"] \arrow[d, phantom, "\cap"] & Z \arrow[dl] \\ X_S & \end{tikzcd}$$

$s_0^* T_{Z/X_S}$ has only pos

then $\exists\ U \to S$ étale, image contains S_0

s.t. $\underline{\underline{S_0}}$ lifts to $X_U \to Z$

Idea: $X_S = Y_{S,[1,q]} / (Y_{S,[q,q]} \cong Y_{S,[1,1]})$

$$\uparrow$$

$Z = Z_{[1,q]} / (Z_{[q,q]} \overset{\varphi_Z}{\cong} Z_{[1,1]})$

can arrange $Z_{[1,q]}$ affinoid, a small ball

all information is in the iso φ_Z:

this preserves zero section over S_0

can arrange that it's "very close" to

linear C by localizing further)

Then do some "Banach fixed point like"

argument to produce φ_Z-invariant sections

of

$$\begin{array}{ccc} & & Z_{S,[1,q]} \\ & \nearrow & \downarrow \\ Y_{S_0,[1,q]} & \subseteq & Y_{S,[1,q]} \end{array}$$

using quantitative version of vanishing $H^1(S_0, s_0^{\#} T_{Z/X_S})$

(lift always exists)

as we assume $Z \to X_S$ smooth

Q: affinoid in LT tower

realize local LLC

away CM pt

near CM pt more and more components

Q: <u>Jacobian</u> criterion

and <u>Jacobian</u>

$x^2+y^2+z^2=0$;

$s^* T_{Z/X_S}$: $0 \to s^* T_{Z/X_S} \to \mathcal{O}(1)^2 \xrightarrow{(2x,2y,2z)} \mathcal{O}(2)$

$\to 0$

Q: ☆ cohomological smooth $\Rightarrow$ formal smooth

$\Leftarrow$?

e.g. $B^+_{dR}/\xi^2 \to \mathbb{A}^1$ is coh. smooth

Q: finite dim of moduli of Shtuka

smooth is not enough

really use geometry of Bun$_G$

Q: qc + smooth $\Rightarrow$ finite dim H^*