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Geometrization of trace formulas. (English summary)
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This paper proposes a geometric framework for applying trace formulas to the various problems in the Langlands program for global function fields.

The word "geometrization" in the title refers to the function-sheaf dictionary of Grothendieck. For an algebraic variety X defined over a finite field k, one can pass from sheaves $\mathcal{F}$ on X (more precisely, sheaves for the étale topology of X with coefficients in $\overline{\mathbb{Q}_\ell}$) to functions $f\colon X(k)\to\overline{\mathbb{Q}_\ell}$ by taking $f(x)$ to be the trace of the Frobenius at x on the stalk $\mathcal{F}_x$. Conversely, interesting functions on $X(k)$ often come from sheaves via this procedure. Thus, one may expect that an equality between two functions f_1 and f_2 on $X(k)$ should come from an isomorphism between two sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ on X.

The trace formula is a fundamental tool in the theory of automorphic forms. It expresses the sum of the traces of an integral operator (given by a Schwartz function f on $G(\mathbb{A}_F)$) on all automorphic representations of $G(\mathbb{A}_F)$ (counted with multiplicity) as a sum of orbital integrals of the same function f. The two sides of the trace formula are usually called the spectral side and the geometric side (or orbital integral side), in the order they appear in the previous sentence.

Trace formulas are often used in the following way. If one would like to establish a relationship between automorphic representations for two groups G and H, one can first try to establish an identity between the orbital integrals on G and H for well-chosen test functions f_G and f_H, and then use trace formulas to get identities on the spectral side.

The idea of geometrization has been systematically applied to the Langlands program for global function fields, starting with Drinfeld, Laumon and then evolving into the geometric Langlands program. However, the authors of this paper seem to be the first to apply geometrization to the study of trace formulas.

The first part of the paper geometrizes the orbital integral side of the trace formula. Let F be the function field of an algebraic curve X over a finite field k. Here the authors restrict to the case when the test function f is in the spherical Hecke algebra at all places. In this case, the orbital integral side of the trace formula is expressed as the cohomology of a complex of sheaves on the moduli stack $\mathcal{M}_G$ of (group-valued) G-Higgs bundles on X. This moduli stack $\mathcal{M}_G$ is closely related to, but not the same as, the Hitchin moduli stack which one of the authors used to prove the Fundamental Lemma [B. C. Ngô, Publ. Math. Inst. Hautes Études Sci. No. 111 (2010), 1–169; MR2653248 (2011h:22011)]. The Hitchin moduli stack is the Lie algebra analog of $\mathcal{M}_G$. Grothendieck's function-sheaf dictionary is applied here, for example, to both $\mathcal{M}_G$ and the moduli stack Bun_G of all G-bundles on X.

Using this interpretation of one side of the trace formula, the authors formulate a precise conjecture relating the cohomology of two such moduli stacks $\mathcal{M}_G$ and $\mathcal{M}_H$, where $G=\mathrm{SL}_2$ and H is a non-split one-dimensional torus over X. A proof of this conjecture will serve as an essential step in establishing Langlands functoriality between G and H, which can be viewed as a test case of the Beyond Endoscopy program proposed in [E. V. Frenkel, R. P. Langlands and B. C. Ngô, Ann. Sci. Math. Québec **34** (2010), no. 2, 199–243; MR2779866 (2012c:11240)].

The second part gives the geometrization of the spectral side of the trace formula. Now all geometric objects are defined over $\mathbb{C}$. Here the authors are inspired by the categorical geometric Langlands conjecture formulated by Beilinson and Drinfeld. Roughly speaking, the spectral side should geometrize to the Hochschild cohomology of certain coherent sheaves on $\mathrm{Loc}_{{}^LG}$, the moduli stack of algebraic ${}^LG$-connections on the same curve X, where ${}^LG$ is the Langlands dual of G. Combining with results from the first part, the authors arrive at a geometrization of the trace formula as an isomorphism between two cohomology groups: Hochschild cohomology of $\mathrm{Loc}_{{}^LG}$ on the one hand and singular cohomology of $\mathcal{M}_G$ on the other (both with respect to appropriate sheaves). This isomorphism should be read as a conjecture, which would follow if the categorical geometric Langlands correspondence was established. Variants in the relative trace formula setting are also discussed in the end.

The paper should be pleasant to read for readers with a bit of background on the geometric Langlands program. It will become a source of inspiration for researchers working on the Langlands program in general.

Reviewed by *Zhiwei Yun*