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A categorical approach to the stable center conjecture. (English, French summaries)

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Let G be a connected reductive group over a non-archimedean local field F. The Bernstein center Z_G is the center of the category of smooth complex representations of $G(F)$, which is in some sense the ring of regular functions on the set of irreducible representations of $G(F)$. Each element of Z_G determines an invariant distribution on $G(F)$. There is a subtle notion of stability for invariant distributions on $G(F)$, which has to do with the fact that two distinct $G(F)$-conjugacy classes in $G(F)$ can become conjugate under $G(\overline{F})$ ($\overline{F}$ is an algebraic closure of F). Let Z_G^{st} be the subspace of Z_G consisting of those elements which are stable as invariant distributions.

The stable Bernstein center conjecture says that Z_G^{st} is a unital subalgebra of Z_G, i.e., as a subspace of invariant distributions, Z_G^{st} should be closed under convolution. A refined version of the conjecture says that if we view Z_G as functions on the set of irreducible representations of $G(F)$, Z_G^{st} should consist exactly of those functions that are constant on L-packets. Here, according to the local Langlands conjecture, the irreducible representations of $G(F)$ are partitioned into finite subsets called L-packets, which are indexed by Galois representations into the dual group ${}^LG$.

The paper under review proposes a geometric way of constructing many (conjecturally all) elements in the depth zero part Z_G^0 of the Bernstein center, when F is a local function field and G is split. This approach opens up the possibility of using geometric and sheaf-theoretic machinery to prove the stability of certain elements in Z_G^0, in the same spirit that geometric methods helped prove the fundamental lemma. As a concrete implementation of this strategy, it is checked in this paper that the unit element in Z_G^0 is stable, a statement which was not known before.

The proposal has its origin in a new construction of character sheaves on a reductive group H as the categorical center of the monoidal category of sheaves on $U\backslash H/U$ (where $U\subset H$ is a maximal unipotent subgroup) [see D. Ben-Zvi and D. Nadler, "The character theory of a complex group", preprint, arXiv:0904.1247; R. Bezrukavnikov, M. Finkelberg and V. Ostrik, Invent. Math. **188** (2012), no. 3, 589–620; MR2917178]. In the current paper the authors carry out an analogous construction when H is replaced with the loop group LG, with (inevitably) a much higher level of technicality. They consider the monoidal category of constructible sheaves on $\mathbf{I}^+\backslash\mathbf{LG}/\mathbf{I}^+$, where $\mathbf{I}^+$ is the pro-unipotent radical of an Iwahori subgroup $\mathbf{I}$ of LG. Then take $\mathcal{Z}_{\mathbf{I}^+}(LG)$ to be the categorical center of the above monoidal category. There is an averaging functor sending an object $\mathcal{B}\in\mathcal{Z}_{\mathbf{I}^+}(LG)$ to an LG-equivariant complex of sheaves $\mathrm{Av}(\mathcal{B})$ on LG, which further carries an action of the affine Weyl group $\widetilde{W}$. Taking the derived invariants of the $\widetilde{W}$-module $\mathrm{Av}(\mathcal{B})$ against the sign representation of $\widetilde{W}$, the resulting object $\mathcal{A}(\mathcal{B})$ on LG is the proposed geometric incarnation of an element in Z_G^0. More precisely, via the sheaf-to-function correspondence, the Frobenius trace at a regular semisimple point $\gamma\in G(F)$ (viewed as a k-point of LG where k is the residue field of F) is the value of a well-defined element $[\mathcal{B}]\in Z_G^0$ at γ.

There are several difficulties new to the current situation that the authors had to overcome: one is that the sheaves they work with live over infinite-dimensional spaces such as LG; another one, which is more essential, is that in order to talk about derived invariants of $\widetilde{W}$-modules, the $\widetilde{W}$-action on $\mathrm{Av}(\mathcal{B})$ needs to be defined at an enhanced categorical level beyond the mere action on the cohomology sheaves of $\mathrm{Av}(\mathcal{B})$. To overcome these difficulties, the authors systematically use the language of stable ∞-categories. In fact, the entirety of sections 1 and 2, which occupy more than a third of the paper, are devoted to setting up categorical foundations for sheaf theory on nice infinite-dimensional ind-schemes and ind-stacks. This part should be of independent interest.

In the simplest case when $\mathcal{B}$ is the unit object in $\mathcal{Z}_{\mathbf{I}^+}(LG)$, the stalk of $\mathrm{Av}(\mathcal{B})$ at a regular semisimple element $\gamma \in LG$ is the homology of the affine Springer fiber Fl_γ, on which the $\widetilde{W}$-action was constructed by G. Lusztig [Transform. Groups **1** (1996), no. 1-2, 83–97; MR1390751]. In this case, the authors show that the corresponding element $z^0 := [\mathcal{B}] \in Z^0_G$ is the projector to the depth zero part Z^0_G of Z_G. In particular, z^0 should be in the stable Bernstein center Z^{st}_G. The main numerical result of this paper (Theorem 4.4.9) confirms that this is indeed the case, at least when z^0 is restricted to regular semisimple elements. The proof is achieved in two steps: first, the authors give a formula for the value of z^0 at a regular semisimple element γ in terms of the Frobenius and $\widetilde{W}$-actions on the homology of the affine Springer fiber Fl_γ (Theorem 4.4.8). Second, they use a theorem of the reviewer [Math. Ann. **359** (2014), no. 3-4, 557–594; MR3231007] on the compatibility of two actions on the homology of Fl_γ to deduce the stability of z^0. Note that the proof of the theorem in [Z. Yun, op. cit.] ultimately uses global methods involving the geometry of Hitchin fibrations.

In section 3.5, there is also an outline of how to get all elements in the depth zero stable Bernstein center. The idea is to start with objects in $\mathcal{Z}_{\mathbf{I}^+}(LG)$ coming from D. Gaitsgory's nearby cycles construction [Invent. Math. **144** (2001), no. 2, 253–280; MR1826370].

Given the technical nature of the subject, the paper is very clearly written. Some of the proofs in section 3 are postponed to a future publication. *Zhiwei Yun*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.