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Epipelagic representations and rigid local systems

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Abstract We construct automorphic representations for quasi-split groups G over the function field $F = k(t)$ one of whose local components is an epipelagic representation in the sense of Reeder and Yu. We also construct the attached Galois representations under the Langlands correspondence. These Galois representations give new classes of conjecturally rigid, wildly ramified ${}^{L}G$-local systems over $\mathbb{P}^1 - \{0, \infty\}$ that generalize the Kloosterman sheaves constructed earlier by Heinloth, Ngô and the author. We study the monodromy of these local systems and compute all examples when G is a classical group.

Keywords Rigid local systems · Vinberg θ-groups · Langlands correspondence

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1 Introduction

1.1 The goal

Let G be a reductive quasi-split group over a local field K as in Sect. 1.4.4. Recently, Reeder and Yu [25] constructed a family of supercuspidal representations of p-adic groups called *epipelagic representations*, generalizing the *simple supercuspidals* constructed earlier by Gross and Reeder [10]. These are supercuspidal representations constructed by compactly inducing certain characters of the pro-p part of a parahoric subgroup $\mathbf{P}$ of G. The construction of epipelagic representations uses the θ-groups $(L_{\mathbf{P}}, V_{\mathbf{P}})$ studied by Vinberg et al.

The main results of this paper are

- Realization of epipelagic representations (for the local function field) as a local component of an automorphic representation π of $G(\mathbb{A}_F)$, where F is the function field $k(t)$. This is done in Proposition 2.11. Here $\pi = \pi(\chi, \phi)$ depends on two parameters χ and ϕ, a character χ of $\widetilde{L}_{\mathbf{P}}(k)$ and a *stable* linear functional $\phi : V_{\mathbf{P}} \to k$.
- Construction of the Galois representation $\rho_\pi : \mathrm{Gal}(F^s/F) \to {}^LG(\overline{\mathbb{Q}}_\ell)$ attached to $\pi = \pi(\chi, \phi)$ under the Langlands correspondence. This is done in Theorem 3.8 and Corollary 3.10. The Galois representation ρ_π can be equivalently thought of as an ℓ-adic ${}^LG$-local systems $\mathrm{Kl}_{{}^LG,\mathbf{P}}(\mathcal{K}, \phi)$ over $\mathbb{P}^1 - \{0, \infty\}$, where ${}^LG$ is the Langlands dual group of G and $\mathcal{K}$ is the incarnation of χ as a rank one character sheaf. We also offer a way to calculate these ${}^LG$-local systems in terms of the Fourier transform (see Proposition 3.12).
- Description of the local monodromy of the local systems $\mathrm{Kl}_{{}^LG,\mathbf{P}}(\mathcal{K}, \phi)$ at 0 (the tame point). The main result is Theorem 4.5 that describes the unipotent monodromy of $\mathrm{Kl}_{{}^LG,\mathbf{P}}(\mathcal{K}, \phi)$ at 0 when G is split and $\mathcal{K}$ is trivial. We also conditionally deduce the cohomological rigidity of $\mathrm{Kl}_{{}^LG,\mathbf{P}}(\mathcal{K}, \phi)$ (see Sect. 5) and make predictions on the monodromy at 0 in general (see Sect. 4.10).
- Computation of the local systems $\mathrm{Kl}_{{}^LG,\mathbf{P}}(\mathcal{K}, \phi)$ when G is a classical group (Sects. 6, 7, 8). In each of these cases, we express the local system $\mathrm{Kl}_{{}^LG,\mathbf{P}}(\mathcal{K}, \phi)$ as the Fourier transform of the direct image complex of an explicit morphism to $V_{\mathbf{P}}$. As a result, we obtain new families of exponential sums (indexed by G and $\mathbf{P}$) generalizing Kloosterman sums; see Corollaries 6.6, 7.6 and 8.7.

1.2 Comparison with [11]

In [11], Heinloth, Ngô and the author considered the case of "simple supercuspidals" of Gross and Reeder. These correspond to the special case $\mathbf{P} = \mathbf{I}$ is an Iwahori subgroup. In [11], we constructed the Kloosterman sheaves $\mathrm{Kl}_{\widehat{G},\mathbf{I}}(\mathcal{K}, \phi)$ and proved properties of their local and global monodromy expected by Frenkel and Gross in [7] for split G.

The general case to be considered in this article exhibits certain interesting phenomena which are not seen in the special case treated in [11].

First, the $\widehat{G}$-local systems $\mathrm{Kl}_{\widehat{G},\mathbf{P}}(\mathcal{K}, \phi)$ on $\mathbb{P}^1 - \{0, \infty\}$ form an algebraic family as the stable linear functional ϕ defining π_∞ varies. In other words, these local systems are obtained from a single "master" $\widehat{G}$-local system $\mathrm{Kl}_{\widehat{G},\mathbf{P}}(\mathcal{K})$ on the larger base $V_{\mathbf{P}}^{*,\mathrm{st}}$ by restriction to various $\mathbb{G}_m$-orbits. Here the base space $V_{\mathbf{P}}^{*,\mathrm{st}}$ is the stable locus of the dual vector space of $V_{\mathbf{P}}$ which is part of Vinberg's θ-groups. The master local system $\mathrm{Kl}_{\widehat{G},\mathbf{P}}(\mathcal{K})$ descends to the stack $[V_{\mathbf{P}}^{*,\mathrm{st}}/L_{\mathbf{P}}]$ (see Lemma 3.9). In the case $\mathbf{P}=\mathbf{I}$ treated in [11], the quotient $[V_{\mathbf{P}}^{*,\mathrm{st}}/L_{\mathbf{P}}]\cong[(\mathbb{P}^1-\{0,\infty\})/ZG]$, and the existence of the master local system does not give extra information in this case. However, for general $\mathbf{P}$, the quotient $[V_{\mathbf{P}}^{*,\mathrm{st}}/L_{\mathbf{P}}]$ has higher dimension, and the existence of the master local system giving all $\mathrm{Kl}_{\widehat{G},\mathbf{P}}(\mathcal{K},\phi)$ at once is stronger than the existence of each of them separately.

Second, when $\mathcal{K}=\mathbf{1}$ is the trivial character sheaf, the local system $\mathrm{Kl}_{\widehat{G},\mathbf{P}}(\mathbf{1},\phi)$ has unipotent tame monodromy at 0 given by a unipotent class $\underline{u}$ in $\widehat{G}^{\sigma,\circ}$ (σ is the pinned automorphism defining G), which only depends on the type of $\mathbf{P}$. On the other hand, the types of $\mathbf{P}$ are in bijection with regular elliptic $\mathbb{W}$-conjugacy classes $\underline{w}$ in $\mathbb{W}\sigma$ (at least when $\mathrm{char}(k)$ is large, see Sect. 2.6). Therefore, our construction gives a map

$$\{\text{regular elliptic } \mathbb{W}\text{-conjugacy classes in } \mathbb{W}\sigma\}\to\{\text{unipotent classes in } \widehat{G}^{\sigma,\circ}\} \quad (1.1)$$

sending $\underline{w}$ to $\underline{u}$ via the intermediate step of an admissible parahoric subgroup $\mathbf{P}$. When G is split, Theorem 4.5 gives an alternative description of this map using Lusztig's theory of cells in affine Weyl groups, and using this we are able to compute the map (1.1) for all types of G in Sect. 4.9 (verified for split G and conjectural in general). In [21], Lusztig defined a map from all conjugacy classes in $\mathbb{W}$ to unipotent conjugacy classes in $\widehat{G}$. One can check case-by-case that this map coincides with the restriction of Lusztig's map to regular elliptic conjugacy classes.

1.3 Open questions

As in the work of Frenkel and Gross [7], we expect that there should be a parallel story when ℓ-adic local systems are replaced with connections on algebraic vector bundles (on varieties over $\mathbb{C}$). In particular, the "master" $\widehat{G}$-local system $\mathrm{Kl}_{\widehat{G},\mathbf{P}}(\mathcal{K})$ should correspond to a $\widehat{G}$-connection over $V_{\mathbf{P},\mathbb{C}}^{*,\mathrm{st}}$. When G is a classical group, the formulae in Sects. 6–8 give descriptions of these connections as Fourier transform of Gauss–Manin connections. Are there simple formulae for these connections in general?

Another problem is to calculate the Euler characteristics (equivalently Swan conductor at ∞) of the local systems $\mathrm{Kl}_{\widehat{G},\mathbf{P}}^{V}(1,\phi)$ for representations V of $\widehat{G}$. Such calculations would give evidence to (and sometimes proofs of) the prediction made by Reeder and Yu about the Langlands parameters of epipelagic representations (see Sect. 2.9). We do one such calculation in the case G is a unitary group; see Proposition 6.8.

While we have a more or less complete picture for the local monodromy of the local systems $\mathrm{Kl}_{\widehat{G},\mathbf{P}}(1,\phi)$, we do not discuss their global monodromy here, i.e., the Zariski closure of the image of the geometric $\pi_1(\mathbb{P}^1-\{0,\infty\})$ in $\widehat{G}$. It can be as small as a finite group as we see in Proposition 6.7(2) when G is an odd unitary group and

$\mathbf{P}$ is a special parahoric subgroup. Does the global monodromy group of $\mathrm{Kl}_{\widehat{G},\mathbf{P}}(1,\phi)$ depend only on $\mathbf{P}$, and if so, how do we read it off from $\mathbf{P}$?

1.4 Notation and conventions

1.4.1 The function field

In this paper, we fix a finite field k and let $F = k(t)$ be the rational function field over k. Places of F are in natural bijection with closed points of $X := \mathbb{P}^1_k$, the set of which is denoted by $|X|$. In particular, we have two places 0 and ∞ of F. For a place $x \in |X|$, let F_x (resp. $\mathcal{O}_x$) be the completed local field (resp. completed local ring) of X at x, and let $k(x)$ be the residue field at x. Let $\mathbb{A}_F = \prod'_{x\in|X|} F_x$ be the ring of adèles of F.

1.4.2 Sheaves

Fix a prime number ℓ different from $\mathrm{char}(k)$. We shall consider constructible $\overline{\mathbb{Q}}_\ell$-complexes over various algebraic stacks over k or $\bar{k}$. All sheaf-theoretic operations are understood as *derived functors*.

1.4.3 The absolute group data

Fix a split reductive group $\mathbb{G}$ over k whose derived group is almost simple. Fix a pinning $\dagger = (\mathbb{B}, \mathbb{T}, \dots)$ of $\mathbb{G}$, where $\mathbb{B}$ is a Borel subgroup of $\mathbb{G}$ and $\mathbb{T} \subset \mathbb{B}$ a split maximal torus. Let $\mathbb{W} = N_{\mathbb{G}}(\mathbb{T})/\mathbb{T}$ be the Weyl group of $\mathbb{G}$. Let $\Phi \subset \mathbb{X}^*(\mathbb{T})$ be the set of roots. Fix a cyclic subgroup $\mathbb{Z}/e\mathbb{Z} \hookrightarrow \mathrm{Aut}^\dagger(\mathbb{G})$ of the pinned automorphism group of $\mathbb{G}$, and denote the image of 1 by $\sigma \in \mathrm{Aut}^\dagger(\mathbb{G})$.

We assume $\mathrm{char}(k)$ is prime to e and that $k^\times$ contains eth roots of unity.

Let $Z\mathbb{G}$ denote the center of $\mathbb{G}$. We also assume that $\mathbb{X}^*(Z\mathbb{G})^\sigma = 0$.

1.4.4 The quasi-split group

Fix a μ_e-cover $\widetilde{X} \to X$ which is totally ramified over 0 and ∞. Then $\widetilde{X}$ is also isomorphic to $\mathbb{P}^1_k$ with affine coordinate $t^{1/e}$. We denote

$$X^\circ := X - \{0,\infty\}; \quad \widetilde{X}^\circ := \widetilde{X} - \{0,\infty\}.$$

The data $\mathbb{G}$ and σ define a quasi-split group scheme G over X° which splits over the μ_e-cover $\widetilde{X}^\circ \to X^\circ$. More precisely, for any $k[t,t^{-1}]$-algebra R, we have $G(R) = \mathbb{G}(R \otimes_{k[t,t^{-1}]} k[t^{1/e}, t^{-1/e}])^{\mu_e}$ where μ_e acts on $t^{1/e}$ by multiplication and on $\mathbb{G}$ via the fixed map $\mu_e \hookrightarrow \mathrm{Aut}^\dagger(\mathbb{G})$. Since $\mathbb{X}^*(Z\mathbb{G})^\sigma = 0$, the center of G does not contain a split torus.

Let $\mathbb{S}$ be the neutral component of $\mathbb{T}^\sigma$. Then $S = \mathbb{S} \otimes_k F$ is a maximal split torus of G over F.

1.4.5 Langlands dual group

Let $\widehat{G}$ denote the reductive group over $\overline{\mathbb{Q}}_\ell$ whose root system is dual to that of $\mathbb{G}$. We also fix a pinning $\dagger$ of $\widehat{G}$, through which we identify $\mathrm{Aut}^\dagger(\widehat{G})$ with $\mathrm{Aut}^\dagger(\mathbb{G})$. We define the Langlands dual group ${}^LG$ of G to be ${}^LG = \widehat{G} \rtimes \mu_e$ where μ_e acts through $\mu_e \hookrightarrow \mathrm{Aut}^\dagger(\mathbb{G}) \cong \mathrm{Aut}^\dagger(\widehat{G})$.

1.4.6 Loop groups

For a local function field K with residue field k and a reductive group H over K, $H(K)$ can be viewed as the set of k-points of a group ind-scheme LH over k: for a k-algebra R, $LH(R)$ is defined to be $H(R\widehat{\otimes}_k K)$. To each parahoric subgroup $\mathbf{P} \subset H(K)$, there is a canonical way to attach a group scheme $\underline{\mathbf{P}} \subset LH$ such that $\underline{\mathbf{P}}(k) = \mathbf{P}$. In the sequel, we shall omit the underline from the notation $\underline{\mathbf{P}}$. When we want to emphasize the dependence on K, we write $L_K G$.

We also need a notion of loop groups varying according to a point $x \in X$. For details we refer to [29, §2.2]. For example, when $x \in X$, we use $L_x G$ (resp. $L_x^+ G$, if $x \in X^\circ$) to denote the loop group (resp. positive loop group) of the group scheme G at x. Then $L_x G$ (resp. $L_x^+ G$) is a group ind-scheme (resp. pro-algebraic group) over $k(x)$, the residue field of x. A parahoric subgroup of $G(F_x)$ is also viewed as a pro-algebraic subgroup of $L_x G$ over $k(x)$. Since G splits over $\widetilde{X}^\circ$, for $\widetilde{x} \in \widetilde{X}^\circ$, we use $L_{\widetilde{x}}\mathbb{G}$ to denote the loop group of the constant group scheme $G \times_{X^\circ} \widetilde{X}^\circ \cong \mathbb{G} \times \widetilde{X}^\circ$ at $\widetilde{x}$.

2 Epipelagic representations and automorphic representations

In this section, we let $K = F_\infty$ denote the local field of F at ∞. We first recall the construction of epipelagic representations of $G(K)$ following Reeder and Yu [25]. We then realize these epipelagic representations as the local components at ∞ of automorphic representations of $G(\mathbb{A}_F)$.

2.1 The affine root system

Using the affine coordinate t on $\mathbb{P}^1$, we write $K = k((t^{-1}))$ and let $K_e = k((t^{-1/e}))$. By construction, $G(K) = \mathbb{G}(K_e)^{\mu_e}$. Therefore, we have the inclusion of the loop groups $LG \subset L_{K_e}\mathbb{G}$.

Consider the apartment $\mathfrak{A}$ in the building of $G(K)$ corresponding to the maximal split torus $S = \mathbb{S} \otimes_k F$. The set Ψ_{aff} of affine roots of $G(K)$ are certain affine functions on $\mathfrak{A}$.

Consider the one-dimensional torus $\mathbb{G}_m^{\mathrm{rot}}$ over k acting on K_e by scaling the uniformizer of $t^{-1/e}$. This action induces an action of $\mathbb{G}_m^{\mathrm{rot}}$ on LG and its Lie algebra $\mathfrak{g}(K)$.

We may identify affine roots Ψ_{aff} as the set of characters of the torus $\mathbb{S} \times \mathbb{G}_m^{\mathrm{rot}}$ that appear in its action on $\mathfrak{g}(K)$. Here $\mathbb{S}$ acts on $\mathfrak{g}(K)$ by the adjoint action and action of $\mathbb{G}_m^{\mathrm{rot}}$ is described in the previous paragraph.

2.2 Affine root subgroup

Let $\Phi_{\mathrm{re}} \subset \Phi_{\mathrm{aff}}$ be the set of real affine root, i.e., those that are nontrivial on $\mathbb{S}$. For each $\alpha \in \Psi_{\mathrm{re}}$, there is a subgroup $LG(\alpha) \subset LG$, isomorphic to the additive group $\mathbb{G}_a$ over k, whose Lie algebra is the α-eigenspace under the action of $\mathbb{S} \times \mathbb{G}_m^{\mathrm{rot}}$. To define it, we write $\alpha = \overline{\alpha} + n\delta$ with $\overline{\alpha} \in \mathbb{X}^*(\mathbb{S})$, $n \in \mathbb{Z}$, and δ the generator of $\mathbb{X}^*(\mathbb{G}_m^{\mathrm{rot}})$. We have a root subgroup $\mathbb{G}_{\overline{\alpha}} \subset \mathbb{G}$. When $2\overline{\alpha}$ is not a root of $\mathbb{G}$, we have $\mathbb{G}_{\overline{\alpha}} \cong \mathbb{G}_a$. In this case, $LG(\alpha)$ is the subgroup of $L_{K_e}\mathbb{G}_{\overline{\alpha}} \cong L_{K_e}\mathbb{G}_a$ whose k-points are $\{ct^{n/e}; c \in k\} \subset K_e$. If $2\overline{\alpha}$ is also a root of $\mathbb{G}$ (this happens only if $\mathbb{G}$ is of type A_{2n} and σ is the nontrivial pinned automorphism), $\mathbb{G}_{\overline{\alpha}}$ can be identified with a Heisenberg group of dimension three, consisting of matrices $\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}$. In this case, $LG(\overline{\alpha} + n\delta)$ is the subgroup of $L_{K_e}\mathbb{G}_{\overline{\alpha}}$ whose k-points consists of $\begin{pmatrix} 1 & ct^{n/2} & 0 \\ & 1 & ct^{n/2} \\ & & 1 \end{pmatrix}$ if n is even and $\begin{pmatrix} 1 & ct^{n/2} & 0 \\ & 1 & -ct^{n/2} \\ & & 1 \end{pmatrix}$ if n is odd.

For a subgroup $\mathbf{H} \subset LG$, we use $\Psi_{\mathrm{re}}(\mathbf{H})$ to denote the set of $\alpha \in \Psi_{\mathrm{re}}$ such that $LG(\alpha) \subset \mathbf{H}$.

2.3 Parahoric subgroups

The Borel subgroup $\mathbb{B} \subset \mathbb{G}$ allows us to define a standard Iwahori subgroup $\mathbf{I} \subset G(K)$ in the following way. We define $\mathbf{I}$ to be the preimage of $\mathbb{B}$ under the homomorphism $G(K) \cap \mathbb{G}(\mathcal{O}_{K_e}) \to \mathbb{G}(\mathcal{O}_{K_e}) \twoheadrightarrow \mathbb{G}$, the last map being given by $\mathcal{O}_{K_e} \twoheadrightarrow k$.

Let $\mathbf{P} \subset G(K)$ be a standard parahoric subgroup (i.e., $\mathbf{I} \subset \mathbf{P}$). Then $\mathbf{P}$ determines a facet $\mathfrak{F}_{\mathbf{P}}$ in $\mathfrak{A}$, and let $\xi \in \mathfrak{A}$ denote its barycenter. Define $m = m(\mathbf{P})$ to be the smallest positive integer such that $\alpha(\xi) \in \frac{1}{m}\mathbb{Z}$ for all affine roots $\alpha \in \Psi_{\mathrm{aff}}$ (see [25, §3.3]).

2.4 Moy-Prasad filtration

Let $\mathbf{P} \supset \mathbf{P}^+ \supset \mathbf{P}^{++}$ be the first three steps in the Moy-Prasad filtration of $\mathbf{P}$. In particular, $\mathbf{P}^+$ is the pro-unipotent radical of $\mathbf{P}$ and $L_{\mathbf{P}} := \mathbf{P}/\mathbf{P}^+$ is the Levi factor of $\mathbf{P}$, a connected reductive group over k. We have

$$\Psi_{\mathrm{re}}(\mathbf{P}) = \{\alpha \in \Psi_{\mathrm{re}} | \alpha(\xi) \geq 0\}; \tag{2.1}$$

$$\Psi_{\mathrm{re}}(\mathbf{P}^+) = \left\{\alpha \in \Psi_{\mathrm{re}} | \alpha(\xi) \geq \frac{1}{m}\right\}; \tag{2.2}$$

$$\Psi_{\mathrm{re}}(\mathbf{P}^{++}) = \{\alpha \in \Psi_{\mathrm{re}} | \alpha(\xi) \geq \frac{2}{m}\}. \tag{2.3}$$

There is a unique section $L_{\mathbf{P}} \hookrightarrow \mathbf{P}$ whose image contains $\mathbb{S} = \mathbb{T}^{\sigma,\circ}$. We shall now understand $L_{\mathbf{P}}$ as a subgroup of $\mathbf{P}$ under this section. The root system of $L_{\mathbf{P}}$ is then identified with the subset $\{\alpha \in \Psi_{\mathrm{re}} | \alpha(\xi) = 0\}$ of Ψ_{aff}. The quotient group

$V_{\mathbf{P}} = \mathbf{P}^+/\mathbf{P}^{++}$ is a vector space over k (viewed as an additive group over k) on which $L_{\mathbf{P}} \rtimes \mathbb{G}_m^{\mathrm{rot}}$ acts.

As shown in [25, Theorem 4.1], the pair $(L_{\mathbf{P}}, V_{\mathbf{P}})$ comes from a $\mathbb{Z}/m\mathbb{Z}$-grading on $\mathrm{Lie}(\mathbb{G})$. The geometric invariant theory of such pairs was studied in depth by Vinberg [27].

Definition 2.5 A standard parahoric subgroup $\mathbf{P}$ of $G(K)$ is called *admissible* if there exists a closed orbit of $L_{\mathbf{P}}$ on the dual space $V_{\mathbf{P}}^*$ with finite stabilizers. Such orbits are called *stable*.

Stable $L_{\mathbf{P}}$-orbits on $V_{\mathbf{P}}^*$ form an open subset $V_{\mathbf{P}}^{*,\mathrm{st}} \subset V_{\mathbf{P}}^*$. Elements in $V_{\mathbf{P}}^{*,\mathrm{st}}$ are called stable linear functionals on $V_{\mathbf{P}}$. For a complete list of admissible parahoric subgroups for various types of G when $\mathrm{char}(k)$ is large, see the tables in [9, §7.1, 7.2].

2.6 Admissible parahorics and regular elliptic numbers

Let $\mathbb{W}' = \mathbb{W} \rtimes \mu_e$ where μ_e acts on $\mathbb{W}$ via its pinned action on $\mathbb{G}$. Springer [26] defined the notion of regular elements for $(\mathbb{W}, \sigma)$. We shall use the slightly different notion of $\mathbb{Z}$-regularity defined by Gross et al. in [9, Definition 1]. An element $w \in \mathbb{W}'$ is $\mathbb{Z}$-*regular* if it permutes the roots Φ freely. See [9, Proposition 1] for the relation between $\mathbb{Z}$-regularity and Springer's notion of regularity. Combining [9, Proposition 1] and [26, Proposition 6.4(iv)], one sees that a $\mathbb{Z}$-regular element in $\mathbb{W}\sigma \subset \mathbb{W}'$ is determined, up to $\mathbb{W}$-conjugacy, by its order.

On the other hand, an element $w \in \mathbb{W}'$ is *elliptic* if $\mathbb{X}^*(\mathbb{T}^{\mathrm{ad}})^w = 0$. The order of a $\mathbb{Z}$-regular elliptic element in $\mathbb{W}\sigma \subset \mathbb{W}'$ is called a *regular elliptic number* of the pair $(\mathbb{W}, \sigma)$. The above discussion says that the assignment $w \mapsto$ order of w is a bijection between $\mathbb{Z}$-regular elliptic elements in $\mathbb{W}\sigma$ up to $\mathbb{W}$-conjugacy and regular elliptic number of the pair $(\mathbb{W}, \sigma)$.

When $\mathbf{P}$ is admissible, $\mathrm{char}(k)$ is not torsion for G and $\mathrm{char}(k) \nmid m(\mathbf{P})$, it was shown in [25, Corollary 5.1] that $m(\mathbf{P})$ is a regular elliptic number of $(\mathbb{W}, \sigma)$. Conversely, for every regular elliptic number m of $(\mathbb{W}, \sigma)$, there is a unique admissible standard parahoric $\mathbf{P}$ such that $m = m(\mathbf{P})$. Combing these facts, one sees that when $\mathrm{char}(k)$ is larger than the twisted Coxeter number h_σ of $(\mathbb{G}, \sigma)$, there are natural bijections between the following three sets

(1) Admissible standard parahorics $\mathbf{P}$ of $G(K)$;
(2) $\mathbb{Z}$-regular elliptic elements $w \in \mathbb{W}\sigma$ up to $\mathbb{W}$-conjugacy;
(3) Regular elliptic numbers m of the pair $(\mathbb{W}, \sigma)$.

For a list of regular elliptic numbers in all types of G, see Sect. 4.9.

Example 2.7 (1) The Iwahori $\mathbf{I}$ is always admissible. It corresponds to the twisted Coxeter number h_σ. In this case $L_{\mathbf{I}} = \mathbb{S} = \mathbb{T}^{\sigma,\circ}$, and $V_{\mathbf{I}}$ is the sum of affine simple root spaces of G.

(2) Let $\sigma \in \mathrm{Out}(\mathbb{G})$ be the involution of the Dynkin diagram which acts as $-w_0$ on $\mathbb{X}_*(\mathbb{T}^{\mathrm{ad}})$, where w_0 be the longest element in $\mathbb{W}$. Then the element $w_0\sigma \in \mathbb{W}\sigma$ acts on $\mathbb{X}_*(\mathbb{T}^{\mathrm{ad}})$ by -1. Hence, $w_0\sigma$ is a $\mathbb{Z}$-regular elliptic element of order 2, and $m = 2$ is a regular elliptic number. The corresponding admissible $\mathbf{P}$ is a maximal

parahoric except in type A_1 and C_n. The Levi quotient $L_{\mathbf{P}} \cong \mathbb{G}^{\theta,\circ}$ where θ is a Chevalley involution on $\mathbb{G}$, and $V_{\mathbf{P}}$ can be identified with the (-1)-eigenspace of θ on Lie$\mathbb{G}$.

From now on, till the end of Sect. 5, we shall fix an admissible standard parahoric subgroup $\mathbf{P}$ of $G(K)$. Let $m = m(\mathbf{P})$ be as defined in Sect. 2.6.

2.8 Epipelagic supercuspidal representations

Fix a nontrivial character $\psi : k = \mathbb{F}_q \to \mathbb{Q}_\ell(\mu_p)^\times$. For a stable functional $\phi \in V_{\mathbf{P}}^{*,\mathrm{st}}(k)$, viewed as a linear map $\phi : V_{\mathbf{P}} \to k$, Reeder–Yu [25, Proposition 2.4] show that the compact induction

$$\mathrm{c}-\mathrm{Ind}_{\mathbf{P}^+}^{G(K)}(\psi \circ \phi)$$

is a finite direct sum of irreducible supercuspidal representations of $G(K)$ (recall the center of $G(K)$ is finite). Its simple summands are called *epipelagic representations* of $G(K)$ attached to the parahoric $\mathbf{P}$ and the stable functional ϕ.

2.9 Expected Langlands parameter

Let $\mathcal{I}_K \lhd W(K^s/K)$ be the inertia group and the Weil group of K. Let π be an epipelagic representation of $G(K)$ attached to $\mathbf{P}$ and a stable linear functional $\phi : V_{\mathbf{P}} \to k$. According the the local Langlands conjecture, there should be a Galois representation $\rho_\pi : W(K^s/K) \to {}^LG(\overline{\mathbb{Q}}_\ell)$ attached to π as the conjectural Langlands parameter. In particular, $\mathcal{I}_K$ acts on the Lie algebra $\widehat{\mathfrak{g}}$ of $\widehat{G}$ by composing ρ_π with the adjoint representation of $\widehat{G}$. Guided by the conjecture in [10] relating the adjoint gamma factors and formal degrees, Reeder and Yu predicted in [25, §7.1] that

(1) $\widehat{\mathfrak{g}}^{\mathcal{I}_K} = 0$;
(2) $\mathrm{Swan}(\widehat{\mathfrak{g}}) = \#\Phi/m$.

If moreover char(k) does not divide the order of $\mathbb{W}$, Reeder and Yu made even sharper predictions: the restriction of ρ_π to $\mathcal{I}_K$ should look like

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{I}_K^+ \arrow[r] \arrow[d] & \mathcal{I}_K \arrow[r] \arrow[d, "\rho_\pi"] & \mathcal{I}_K^t \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \widehat{T} \arrow[r] & N_{{}^LG}(\widehat{T}) \arrow[r] & \mathbb{W}' \arrow[r] & 1
\end{tikzcd}$$

such that a generator of the tame inertia $\mathcal{I}_K^t$ maps to a regular elliptic element $w \in \mathbb{W}\sigma$ of order m.

2.10 Realization in automorphic representations

Recall that $K = F_\infty$ and now we denote $\mathbf{P}$ by $\mathbf{P}_\infty$. Let $\mathbf{I}_0 \subset G(F_0)$ be the Iwahori subgroup corresponding to the opposite Borel $\mathbb{B}^{\mathrm{opp}}$ of $\mathbb{G}$, in the same way that $\mathbf{I} = \mathbf{I}_\infty$ was constructed from $\mathbb{B}$. Let $\mathbf{P}_0 \subset G(F_0)$ be the parahoric subgroup containing $\mathbf{I}_0$ of the same type as $\mathbf{P}_\infty$. Let $\widetilde{\mathbf{P}}_0$ be the normalizer of $\mathbf{P}_0$ in $G(F_0)$ and $\Omega_{\mathbf{P}} = \widetilde{\mathbf{P}}_0/\mathbf{P}_0$ a finite group, viewed as a constant group scheme over k. The Levi quotient of $\mathbf{P}_0$ is also identified with $L_{\mathbf{P}}$. Let $\widetilde{L}_{\mathbf{P}} = \widetilde{\mathbf{P}}_0/\mathbf{P}_0^+$, which is an extension of $\Omega_{\mathbf{P}}$ by $L_{\mathbf{P}}$.

Let $L^{\mathrm{sc}}_{\mathbf{P}}$ be the simply connected cover of the derived group of $L_{\mathbf{P}}$. Let $L_{\mathbf{P}}(k)' = \mathrm{Im}(L^{\mathrm{sc}}_{\mathbf{P}}(k) \to L_{\mathbf{P}}(k))$. Fix a character $\chi : \widetilde{L}_{\mathbf{P}}(k)/L_{\mathbf{P}}(k)' \to \overline{\mathbb{Q}}_\ell^\times$. Also fix a *stable* linear functional $\phi : V_{\mathbf{P}} \to k$.

Now we try to classify automorphic representations $\pi = \otimes'_{x\in|X|}\pi_x$ of $G(\mathbb{A}_F)$ satisfying

- π_x is unramified for $x \neq 0, \infty$;
- π_0 has an eigenvector under $\widetilde{\mathbf{P}}_0$ on which it acts through χ via the quotient $\widetilde{L}_{\mathbf{P}}(k)$;
- π_∞ has an eigenvector under $\mathbf{P}^+_\infty$ on which it acts through $\psi \circ \phi$ (i.e., π_∞ is an epipelagic supercuspidal representation attached to $\mathbf{P}^+_\infty$ and the stable functional ϕ).

Proposition 2.11 *There is a unique automorphic representation $\pi = \pi(\chi, \phi)$ of $G(\mathbb{A}_F)$ satisfying all the above conditions. Moreover, we have*

(1) *π is cuspidal and appears with multiplicity one in the automorphic spectrum of G.*
(2) *Both eigenspaces $\pi_0^{(\widetilde{\mathbf{P}}_0,\chi)}$ and $\pi_\infty^{(\mathbf{P}^+_\infty,\psi\circ\phi)}$ are one-dimensional.*

The proof will occupy Sects. 2.12–2.15. The idea of the proof goes as follows. Inside each π as above, there is a subspace consisting of vectors that are fixed by $G(\mathcal{O}_x)$ for $x \neq 0, \infty$ and eigen under $\widetilde{\mathbf{P}}_0$ and $\mathbf{P}^+_\infty$. Such vectors can be realized as functions on a double coset space of $G(\mathbb{A}_F)$ satisfying certain eigen properties. We analyze this double coset in details and show that such functions are unique up to a scalar.

2.12 Functions on a double coset

Consider the vector space of $\overline{\mathbb{Q}}_\ell$-valued functions

$$\mathcal{A} := \mathrm{Fun}\left(G(F)\backslash G(\mathbb{A}_F)/\prod_{x\neq 0,\infty} G(\mathcal{O}_x)\right)^{(\widetilde{\mathbf{P}}_0,\chi)\times(\mathbf{P}^+_\infty,\psi\circ\phi)}.$$

Here the superscript means taking the eigenspace under $\widetilde{\mathbf{P}}_0 \times \mathbf{P}^+_\infty$ on which $\widetilde{\mathbf{P}}_0$ acts through χ and $\mathbf{P}^+_\infty$ acts through $\psi \circ \phi$. We first show that $\dim \mathcal{A} = 1$.

Let $\widetilde{\Gamma}_0 = G(k[t, t^{-1}]) \cap \widetilde{\mathbf{P}}_0$. Let Γ'_0 be the preimage of $L_{\mathbf{P}}(k)' = \mathrm{Im}(L^{\mathrm{sc}}_{\mathbf{P}}(k) \to L_{\mathbf{P}}(k))$ under $\widetilde{\Gamma}_0 \to \widetilde{\mathbf{P}}_0 \to \widetilde{L}_{\mathbf{P}}(k)$. We also view $\widetilde{L}_{\mathbf{P}}$ as a subgroup of $\widetilde{\Gamma}_0$ and view

$L_{\mathbf{P}}(k)'$ as a subgroup of Γ'_0. Let $W_{\mathbf{P}}$ be the Weyl group of $L_{\mathbf{P}}$ with respect to $\mathbb{S}$, identified with a subgroup of the extended affine Weyl group $\widetilde{W}$ of $G(F_\infty)$. Let $\widetilde{W}_{\mathbf{P}} = N_{\widetilde{W}}(W_{\mathbf{P}})$, which is an extension of $\Omega_{\mathbf{P}}$ by $W_{\mathbf{P}}$.

A parahoric variant of [11, Proposition 1.1] gives an equality between double cosets

$$G(F)\backslash G(\mathbb{A}_F)/\left(\widetilde{\mathbf{P}}_0 \times \prod_{x\neq 0,\infty} G(\mathcal{O}_x) \times \mathbf{P}^+_\infty\right) = \widetilde{\Gamma}_0\backslash G(F_\infty)/\mathbf{P}^+_\infty \tag{2.4}$$

and the Birkhoff decomposition

$$G(F_\infty) = \bigsqcup_{\widetilde{w}\in \widetilde{W}_{\mathbf{P}}\backslash \widetilde{W}/W_{\mathbf{P}}} \widetilde{\Gamma}_0 \widetilde{w} \mathbf{P}_\infty.$$

More details about this decomposition will be explained in Sect. 3.2 using a geometric interpretation of the left side of (2.4) in terms of moduli stack of G-bundles over X.

2.13 Uniqueness of automorphic function

Recall from Sect. 2.2 the notation $\Psi_{\mathrm{re}}(\mathbf{H})$ for a subgroup $\mathbf{H}$ of $G(F_\infty)$. We have

$$\Psi_{\mathrm{re}}(\Gamma'_0) = \Psi_{\mathrm{re}}(\Gamma_0) = \{\alpha \in \Psi_{\mathrm{re}} | \alpha(\xi) \leq 0\}.$$

We consider the decomposition of $V_{\mathbf{P}}$ into weight spaces under $\mathbb{S} \times \mathbb{G}_m^{\mathrm{rot}}$. By (2.2) and (2.3), the composition

$$\bigoplus_{\alpha\in\Psi_{\mathrm{re}},\alpha(\xi)=\frac{1}{m}} LG(\alpha) \hookrightarrow \mathbf{P}^+_\infty \to V_{\mathbf{P}},$$

is an embedding as part of the weight decomposition under $\mathbb{S}\times\mathbb{G}_m^{\mathrm{rot}}$; the other weights that appear in $V_{\mathbf{P}}$ are imaginary roots, i.e., those trivial on $\mathbb{S}$.

Suppose f is nonzero on the double coset $\widetilde{\Gamma}_0\widetilde{w}\ell\mathbf{P}^+_\infty$ for some $\widetilde{w} \in \widetilde{W}$ and $\ell \in L_{\mathbf{P}}(k)$. Then f is left invariant by Γ'_0 and eigen under the translation of $\mathbf{P}^+_\infty$ via the character $\mathbf{P}^+_\infty \to V_{\mathbf{P}} \xrightarrow{\psi\circ\phi} \overline{\mathbb{Q}}_\ell$. In particular, for any $\alpha \in \Psi_{\mathrm{re}}$ with $\alpha(\xi) = \frac{1}{m}$ and any $u_\alpha \in LG(\alpha)$, we have

$$f(\widetilde{w}u_\alpha\ell) = f(\widetilde{w}\ell\mathrm{Ad}(\ell^{-1})u_\alpha) = \psi\circ\phi(\mathrm{Ad}(\ell^{-1})u_\alpha)\cdot f(\widetilde{w}\ell). \tag{2.5}$$

Here, $\mathrm{Ad}(\ell^{-1})u_\alpha \in V_{\mathbf{P}}$; hence, it makes sense to evaluate ϕ on $\mathrm{Ad}(\ell^{-1})u_\alpha$. Suppose further that $(\widetilde{w}\alpha)(\xi) \leq 0$, hence $LG(\widetilde{w}\alpha) \subset \Gamma'_0$ and in particular $\mathrm{Ad}(\widetilde{w})u_\alpha \in \Gamma'_0$, then we have

$$f(\widetilde{w}u_\alpha\ell) = f(\mathrm{Ad}(\widetilde{w})u_\alpha\widetilde{w}\ell) = f(\widetilde{w}\ell). \tag{2.6}$$

Combining (2.5) and (2.6), we see that $\psi \circ \phi(\mathrm{Ad}(\ell^{-1})u_\alpha) = 1$ for all $u_\alpha \in LG(\alpha)$; i.e., the linear function $\phi' = \ell \cdot \phi$ annihilates $LG(\alpha)$ for all $\alpha \in \Psi_{\mathrm{re}}$ satisfying $\alpha(\xi) = \frac{1}{m}$ and $\widetilde{w}\alpha(\xi) \leq 0$.

For an affine root β with $\beta(\xi) = \frac{1}{m}$, the condition $\widetilde{w}\beta(\xi) \leq 0$ is equivalent to $\widetilde{w}\beta(\xi) < \beta(\xi)$, or in other words, $\langle \overline{\beta}, \widetilde{w}^{-1}\xi - \xi \rangle < 0$, where $\overline{\beta}$ is the image of β in $\mathbb{X}^*(\mathbb{S})$. Such an affine root is automatically real.

Suppose $\widetilde{w}^{-1}\xi \neq \xi$, let $\lambda = m(\widetilde{w}^{-1}\xi - \xi) \in \mathbb{X}_*(\mathbb{S})$. The action of $\lambda(\mathbb{G}_m) \subset \mathbb{S}$ on $V_{\mathbf{P}}$ decomposes $V_{\mathbf{P}}$ into the direct sum of negative weight space $V_{\mathbf{P}}(< 0)$ and nonnegative weight space $V_{\mathbf{P}}(\geq 0)$ and likewise for the dual space $V_{\mathbf{P}}^*$. The above discussion shows that $\langle \phi', V_{\mathbf{P}}(< 0) \rangle = 0$. Therefore, $\phi' \in V_{\mathbf{P}}^*(\leq 0)$. However, this means that $\phi'' := \lim_{t \to \infty} \lambda(t) \cdot \phi'$ exists, and its stabilizer under $L_{\mathbf{P}}$ contains the nontrivial torus $\lambda(\mathbb{G}_m)$. Since $L_{\mathbf{P}} \cdot \phi = L_{\mathbf{P}} \cdot \phi'$ is a closed orbit, $\phi'' \in L_{\mathbf{P}} \cdot \phi$. However, the stability condition forces ϕ'' to have finite stabilizer under $L_{\mathbf{P}}$. This is a contradiction. The conclusion is that $\widetilde{w}^{-1}\xi = \xi$.

Those $\widetilde{w}$ with $\widetilde{w}^{-1}\xi = \xi$ are precisely those in $\widetilde{W}_{\mathbf{P}}$. Therefore, f, as a function on $G(F_\infty)$, is supported on the unit coset $\widetilde{\Gamma}_0 \mathbf{P}_\infty = \widetilde{\Gamma}_0 \times \mathbf{P}_\infty^+$. The $(\widetilde{\mathbf{P}}_0, \chi)$-eigen property of f as a function on $G(\mathbb{A}_F)$ implies that it is left $(\widetilde{\Gamma}_0, \chi)$-eigen as a function on $G(F_\infty)$. Together with the right $(\mathbf{P}_\infty^+, \psi \circ \phi)$-eigen property, the function f is unique up to a scalar. We have shown that $\dim \mathcal{A} = 1$.

2.14 Cuspidality

Next we check that $\mathcal{A}$ consists of cuspidal functions. In fact, for any $f \in \mathcal{A}$, $g \in G(\mathbb{A}_F)$ and any proper F-parabolic $P \subset G$ with unipotent radical U_P, the constant term $\int_{U_P(F)\backslash U_P(\mathbb{A}_F)} f(ng)dn$ can be written as a finite sum of the form $h(g_\infty) := \int_{\Gamma\backslash U_P(F_\infty)} f(g^\infty, n_\infty g_\infty)dn_\infty$, for some discrete subgroup $\Gamma \subset U_P(F_\infty)$ and $g = (g^\infty, g_\infty)$ where $g_\infty \in G(F_\infty)$, $g^\infty \in G(\mathbb{A}_F^\infty)$. The function $h : G(F_\infty) \to \overline{\mathbb{Q}}_\ell$ defined above is left $U_P(F_\infty)$-invariant and right $(\mathbf{P}_\infty^+, \psi \circ \phi)$-eigen, hence induces a $U_P(F_\infty)$-invariant map $c_h : \mathrm{c}-\mathrm{Ind}_{\mathbf{P}_\infty^+}^{G(F_\infty)}(\psi^{-1} \circ \phi) \to \overline{\mathbb{Q}}_\ell$ by $h' \mapsto \int_{G(F_\infty)/\mathbf{P}_\infty^+} h(x)h'(x^{-1})dx$. However, by Reeder and Yu [25, Proposition 2.4], $\mathrm{c}-\mathrm{Ind}_{\mathbf{P}_\infty^+}^{G(F_\infty)}(\psi^{-1} \circ \phi)$ is a finite direct sum of supercuspidal representations, hence has zero Jacquet module with respect to P. Therefore, $c_h = 0$ and h must be the zero function. This implies that the constant term $\int_{U_P(F)\backslash U_P(\mathbb{A}_F)} f(ng)dn$ must be zero.

2.15 Finish of the proof of Proposition 2.11

Since cuspidal functions belong to the discrete spectrum, we have

$$\dim \mathcal{A} = \sum_{\pi} m(\pi) \dim \pi_0^{(\widetilde{\mathbf{P}}_0, \chi)} \dim \pi_\infty^{(\mathbf{P}_\infty^+, \psi\circ\phi)}. \tag{2.7}$$

where the sum is over isomorphism classes of π satisfying the conditions in Sect. 2.10, and $m(\pi)$ is the multiplicity that π appears in the automorphic spectrum of $G(\mathbb{A}_F)$.

Since $\dim \mathcal{A} = 1$, the right side of (2.7) only has one nonzero term, in which all factors are equal to one. This finishes the proof of Proposition 2.11.

Remark 2.16 Not all epipelagic representations are realized as a local component of π as in Proposition 2.11. By [25, Proposition 2.4], epipelagic representations of $G(F_\infty)$ with $(\mathbf{P}_\infty^+, \psi \circ \phi)$ eigenvectors are indexed by characters ρ of the finite abelian group $\mathrm{Stab}_{\widetilde{L}_\mathbf{P}}(\phi)(k)$. However, only those ρ that can be extended to a character of $\widetilde{L}_\mathbf{P}(k)$ appear as a local component of $\pi(\chi, \phi)$ for some χ as in Proposition 2.11.

By the proof of Proposition 2.11, we see that any nonzero function $f \in \mathcal{A}$ (which is unique up to a scalar by the previous proposition) has to be a simultaneous eigenfunction for the spherical Hecke algebras at $x \neq 0$ or ∞. One can use this fact to calculate the Satake parameters of π_x. In the next section, we shall do this in a more geometric way and construct the ${}^L G$-local system attached to π under the global Langlands correspondence.

3 Generalized Kloosterman sheaves

We keep the notations from Sect. 2.10. In this section, we will construct generalized Kloosterman sheaves as Galois representations attached to the automorphic representations $\pi(\chi, \phi)$ in Proposition 2.11. The construction uses ideas from the geometric Langlands correspondence.

3.1 Moduli stacks of G-bundles

Let $\mathrm{Bun}_G(\mathbf{P}_0^+, \mathbf{P}_\infty^{++})$ be the moduli stack of G-bundles over $X = \mathbb{P}^1$ with $\mathbf{P}_0^+$-level structures at 0 and $\mathbf{P}_\infty^{++}$-level structures at ∞. This is an algebraic stack over k. For the construction of moduli stacks with parahoric level structures, such as $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty)$; see [28, §4.2]; the moduli stack $\mathrm{Bun}_G(\mathbf{P}_0^+, \mathbf{P}_\infty^{++})$ is an $L_\mathbf{P} \times (\mathbf{P}_\infty/\mathbf{P}_\infty^{++})$-torsor over $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty)$. In the sequel, we abbreviate $\mathrm{Bun} := \mathrm{Bun}_G(\mathbf{P}_0^+, \mathbf{P}_\infty^{++})$. The trivial G-bundle with the tautological level structures at 0 and ∞ gives a base point $\star \in \mathrm{Bun}(k)$.

There is an action of $\widetilde{L}_\mathbf{P} \times V_\mathbf{P}$ on Bun because $\widetilde{\mathbf{P}}_0$ normalizes $\mathbf{P}_0^+$ and $\mathbf{P}_\infty^+$ normalizes $\mathbf{P}_\infty^{++}$.

Let $\mathrm{Bun}_G(\widetilde{\mathbf{P}}_0, \mathbf{P}_\infty^+)$ be the quotient of Bun by the $\widetilde{L}_\mathbf{P} \times V_\mathbf{P}$-action. If we only quotient Bun by the action of $L_\mathbf{P} \times V_\mathbf{P}$, the resulting stack is $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty^+)$, the moduli stack of G-bundles over X with $\mathbf{P}_0$-level structures at 0 and $\mathbf{P}_\infty^+$-level structures at ∞. By [11, Proposition 1.1(5)], the connected components of $\mathrm{Bun}_G(\mathbf{I}_0, \mathbf{I}_\infty^+)$ are canonically indexed by $\Omega := \widetilde{W}/W_{\mathrm{aff}}$, where W_{aff} is the affine Weyl group generated by the affine simple reflections. Since $\mathrm{Bun}_G(\mathbf{I}_0, \mathbf{I}_\infty^+) \to \mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty^+)$ has connected fibers, the connected components of $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty^+)$ are also indexed by Ω. Since the map $\widetilde{\mathbf{P}}_0/\mathbf{P}_0 = \Omega_\mathbf{P} \to \Omega$ is an injection, we have $\mathrm{Bun}_G(\widetilde{\mathbf{P}}_0, \mathbf{P}_\infty^+) \cong \mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty^+)/\Omega_\mathbf{P}$; i.e., $\mathrm{Bun}_G(\widetilde{\mathbf{P}}_0, \mathbf{P}_\infty^+)$ is obtained from $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty^+)$ by identifying certain connected components.

3.2 Birkhoff decomposition

The parahoric version of the Birkhoff decomposition [29, §3.2.2] gives a decomposition

$$\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty^+)(k) \cong \Gamma_0\backslash G(F_\infty)/\mathbf{P}_\infty^+ = \bigsqcup_{w\in W_{\mathbf{P}}\backslash \widetilde{W}/W_{\mathbf{P}}} \Gamma_0\backslash(\Gamma_0\widetilde{w}\mathbf{P}_\infty)/\mathbf{P}_\infty^+. \tag{3.1}$$

We may also identify $\Omega_{\mathbf{P}}$ with $\widetilde{W}_{\mathbf{P}}/W_{\mathbf{P}}$, where $\widetilde{W}_{\mathbf{P}}$ is the normalizer of $W_{\mathbf{P}}$ in $\widetilde{W}$. Then $\Omega_{\mathbf{P}}$ acts on the left side of (3.1) by permuting the connected components of $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty^+)$, and this action transferred to the right side acts on the double cosets $W_{\mathbf{P}}\backslash\widetilde{W}/W_{\mathbf{P}}$ by left translation of $\Omega_{\mathbf{P}} = \widetilde{W}_{\mathbf{P}}/W_{\mathbf{P}}$. Taking quotient by $\Omega_{\mathbf{P}}$ on both sides, we obtain

$$\mathrm{Bun}_G(\widetilde{\mathbf{P}}_0, \mathbf{P}_\infty^+)(k) \cong \widetilde{\Gamma}_0\backslash G(F_\infty)/\mathbf{P}_\infty^+ = \bigsqcup_{w\in \widetilde{W}_{\mathbf{P}}\backslash \widetilde{W}/W_{\mathbf{P}}} \widetilde{\Gamma}_0\backslash(\widetilde{\Gamma}_0\widetilde{w}\mathbf{P}_\infty)/\mathbf{P}_\infty^+. \tag{3.2}$$

Moreover, the same is true when k is replaced by its algebraic closure (and with $G(F_\infty)$, $\mathbf{P}_\infty$, etc., base changed to $\bar{k}$). By [29, Lemma 3.1], the unit double coset $\widetilde{w} = 1$ in (3.1) is an open substack of $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}_\infty^+)$. This open substack is in fact a point with trivial stabilizer, and this point is also the image of $\star$. The preimage U of this open point in Bun is then a $\widetilde{L}_{\mathbf{P}} \times V_{\mathbf{P}}$-torsor trivialized by the base point $\star$, giving an open immersion

$$j : U \cong \widetilde{L}_{\mathbf{P}} \times V_{\mathbf{P}} \hookrightarrow \mathrm{Bun}.$$

3.3 Sheaves on Bun

We fix a rank one character sheaf $\mathcal{K}$ on $\widetilde{L}_{\mathbf{P}}$. This is a rank one local system on $\widetilde{L}_{\mathbf{P}}$ together with an isomorphisms $\mu^*\mathcal{K} \cong \mathcal{K} \boxtimes \mathcal{K}$ (where $\mu : \widetilde{L}_{\mathbf{P}} \times \widetilde{L}_{\mathbf{P}} \to \widetilde{L}_{\mathbf{P}}$ is the multiplication) and $\epsilon^*\mathcal{K} \cong \overline{\mathbb{Q}}_\ell$ (where ϵ : Spec$k \hookrightarrow \widetilde{L}_{\mathbf{P}}$ is the identity element) satisfying the usual cocycle relations. For details see [30, Appendix A]. The sheaf-to-function correspondence attaches to $\mathcal{K}$ a character $\chi : \widetilde{L}_{\mathbf{P}}(k) \to \overline{\mathbb{Q}}_\ell^\times$. By [30, Theorem A.3.8], the characters χ arising from character sheaves are exactly those that are trivial on the image of $L_{\mathbf{P}}^{\mathrm{sc}}(k)$. However, when $\widetilde{L}_{\mathbf{P}}$ is disconnected, $\mathcal{K}$ carries more information than its associated character χ.

Let S be a scheme and $s : S \to V_{\mathbf{P}}^*$ be a morphism. We also consider $V_{\mathbf{P}} \times S$ as a constant additive group over S, over which we have a rank one local system $\mathrm{AS}_S := (\mathrm{id} \times s)^*\langle,\rangle^*\mathrm{AS}_\psi$ where $\langle,\rangle : V_{\mathbf{P}} \times V_{\mathbf{P}}^* \to \mathbb{G}_a$ is the natural pairing and AS_ψ is the rank one Artin–Schreier local system on $\mathbb{G}_a$ given by the additive character ψ. The local system AS_S is a character sheaf over S by which we mean the following. Let $a : V_{\mathbf{P}} \times V_{\mathbf{P}} \times S \to V_{\mathbf{P}} \times S$ be the addition map in the first two variables, and let $p_{13}, p_{23} : V_{\mathbf{P}} \times V_{\mathbf{P}} \times S \to V_{\mathbf{P}} \times S$ be the projections onto the factors indicated in the subscripts. Let $0_S : \{0\} \times S \hookrightarrow V_{\mathbf{P}} \times S$ be the inclusion. Then there are isomorphisms $a^*\mathrm{AS}_S \cong p_{13}^*\mathrm{AS}_S \otimes_S p_{23}^*\mathrm{AS}_S$ and $0_S^*\mathrm{AS}_S \cong \overline{\mathbb{Q}}_\ell$ satisfying the usual cocycle relations.

For any scheme S over $V^*_{\mathbf{P}}$, we consider the product $\mathrm{Bun} \times S$, on which the algebraic group $\widetilde{L}_{\mathbf{P}}$ and the constant group scheme $V_{\mathbf{P}} \times S$ act. The derived category of $(\widetilde{L}_{\mathbf{P}}, \mathcal{K})$-equivariant and $(V_{\mathbf{P}} \times S, \mathrm{AS}_S)$-equivariant $\overline{\mathbb{Q}}_\ell$-complexes on $\mathrm{Bun} \times S$ is defined. We denote this category by $\mathcal{D}(\mathcal{K}, S)$.

For $? =!$ or $*$, let

$$A_?(\mathcal{K}, S) = (j \times \mathrm{id}_S)_?(\mathcal{K} \boxtimes \mathrm{AS}_S) \in \mathcal{D}(\mathcal{K}, S).$$

Lemma 3.4 *Let S be a scheme of finite type over $V^{*,\mathrm{st}}_{\mathbf{P}}$.*

(1) *Any object $A \in \mathcal{D}(\mathcal{K}, S)$ has vanishing stalks outside $U \times S$. In particular, the canonical map $A_!(\mathcal{K}, S) \to A_*(\mathcal{K}, S)$ is an isomorphism. We shall denote them by $A(\mathcal{K}, S)$.*

(2) *The functor*

$$\begin{aligned} D^b_c(S) &\to \mathcal{D}(\mathcal{K}, S) \\ C &\mapsto (j \times \mathrm{id}_S)_!((\mathcal{K} \boxtimes \mathrm{AS}_S) \otimes \mathrm{pr}_S^* C) \end{aligned}$$

is an equivalence of categories. Here $\mathrm{pr}_S : U \times S \to S$ *is the projection.*

Proof (2) is an immediate consequences of (1). To show (1), it suffices to check the vanishing property over each closed point of S. Therefore, we may assume $S = \mathrm{Spec}(k')$ for some finite extension k' of k, which corresponds to a point $\phi \in V^{*,\mathrm{st}}(k')$. We base change the situation to k' without changing notation, and in particular we assume $k = k'$. Any character sheaf on $L_{\mathbf{P}}$ is trivial when pulled back to $L^{\mathrm{sc}}_{\mathbf{P}}$; see [30, Theorem A.3.8]. Let $\Gamma^{\mathrm{sc}}_0 = \Gamma_0 \times_{L_{\mathbf{P}}} L^{\mathrm{sc}}_{\mathbf{P}}$, then the character sheaf $\mathcal{K}$ is trivial on Γ^{sc}_0. By the Birkhoff decomposition (3.2), we need to show that for any $\widetilde{w} \notin \widetilde{W}_{\mathbf{P}}$ and any complex of sheaves $\mathcal{F}$ on $Y_{\widetilde{w}} = \widetilde{\Gamma}_0 \widetilde{w} \mathbf{P}_\infty / \mathbf{P}^{++}_\infty$ that are left-Γ^{sc}_0-equivariant and right $(V_{\mathbf{P}}, \phi^* \mathrm{AS}_\psi)$-equivariant must be zero. The argument is a straightforward sheaf-theoretic analog of the argument given in Proposition 2.11. Alternatively, we may deduce that $\mathcal{F} = 0$ from the proof of Proposition 2.11. In fact, since each cohomology sheaf $H^i \mathcal{F}$ also has the same equivariance properties, we may reduce to the case that $\mathcal{F}$ is a sheaf concentrated in degree zero. For any finite extension k'/k, the trace of Frobenius acting on stalks of $\mathcal{F}$ gives a function $f_{k'} : Y_{\widetilde{w}}(k') \to \overline{\mathbb{Q}}_\ell$ that is left $\Gamma^{\mathrm{sc}}_0(k')$-invariant and right $(V_{\mathbf{P}}(k'), \psi \circ \mathrm{Tr}_{k'/k} \circ \phi)$-eigen. In Sect. 2.13, we have shown that such a function $f_{k'}$ must be zero. This being true for all finite extensions k'/k, we conclude that the sheaf $\mathcal{F}$ is identically zero. □

3.5 The Hecke stack

For more details on the Satake equivalence and Hecke operators, we refer to [11, §2.3, 2.4]. In particular, our Satake category consists of weight zero complexes as normalized in [11, Remark 2.10] (note this involves the choice of a half Tate twist). Here we only set up the notation in order to state our main result.

The Hecke correspondence Hk for $\mathrm{Bun} = \mathrm{Bun}(\mathbf{P}^+_0, \mathbf{P}^{++}_\infty)$ classifies $(x, \mathcal{E}, \mathcal{E}', \tau)$ where $x \in \widetilde{X}^\circ$, $\mathcal{E}, \mathcal{E}' \in \mathrm{Bun}$ and $\tau : \mathcal{E}|_{\widetilde{X} - \{x\}} \xrightarrow{\sim} \mathcal{E}'|_{\widetilde{X} - \{x\}}$ is an isomorphism of G-

torsors preserving the level structures at 0 and ∞. Let $\pi : \mathrm{Hk} \to \widetilde{X}^\circ$ be the morphism that remembers only x, and let Hk_x be the fiber of π over $x \in \widetilde{X}^\circ$.

For a dominant coweight $\lambda \in \mathbb{X}_*(\mathbb{T})$, we have a substack $\mathrm{Hk}_{\leq\lambda}$ where the relative position of τ at x is required to be bounded by λ. Let $\mathrm{Gr}_{\widetilde{X}^\circ}$ be the Beilinson-Drinfeld Grassmannian for the constant group $\mathbb{G}$ over $\widetilde{X}^\circ$. For a geometric point $x \in \widetilde{X}^\circ$, the fiber of $\mathrm{Gr}_{\widetilde{X}^\circ}$ over x is the affine Grassmannian $\mathrm{Gr}_x = L_x\mathbb{G}/L_x^+\mathbb{G}$. For each positive integer N, let $\mathbf{K}_{\widetilde{X}^\circ,N}$ be the restriction of scalars $\mathrm{Res}_{\widetilde{X}^\circ}^{J_N}\mathbb{G}$ where $J_N \to \widetilde{X}^\circ$ is the scheme of N-jets on $\widetilde{X}^\circ$. For a geometric point $x \in \widetilde{X}^\circ$, the fiber of $\mathbf{K}_{\widetilde{X}^\circ,N}$ over x is the quotient $\mathbf{K}_{x,N}$ of $L_x^+\mathbb{G}$ by a congruence subgroup. For N sufficiently large (depending on λ), the left action of $L_x\mathbb{G}$ on Gr_x factors through to $\mathbf{K}_{x,N}$, and we get an action of $\mathbf{K}_{\widetilde{X}^\circ,N}$ on $\mathrm{Gr}_{\widetilde{X}^\circ}$ by fiberwise left multiplication. By [13, Eq. (1.10) in Lemme 1.12], there is a morphism

$$\mathrm{ev}_{\leq\lambda} : \mathrm{Hk}_{\leq\lambda} \to [\mathbf{K}_{\widetilde{X}^\circ,N}\backslash\mathrm{Gr}_{\widetilde{X}^\circ,\leq\lambda}]. \tag{3.3}$$

Over the a geometric point $x \in \widetilde{X}^\circ$, this map $\mathrm{ev}_x : \mathrm{Hk}_{x,\leq\lambda} \to [\mathbf{K}_{x,N}\backslash\mathrm{Gr}_{x,\leq\lambda}]$ records the relative position of the isomorphism $\tau : \mathcal{E}|_{\widetilde{X}-\{x\}} \xrightarrow{\sim} \mathcal{E}'|_{\widetilde{X}-\{x\}}$ near x.

In our case, we have $\widetilde{X}^\circ \cong \mathbb{G}_m$. Using the simple transitive $\mathbb{G}_m$-action on $\widetilde{X}^\circ$, the fibration $[\mathbf{K}_{\widetilde{X}^\circ,N}\backslash\mathrm{Gr}_{\widetilde{X}^\circ,\leq\lambda}] \to \widetilde{X}^\circ$ can be trivialized and we may identify it with $\widetilde{X}^\circ \times [\mathbf{K}_N\backslash\mathrm{Gr}_{\leq\lambda}]$, where $\mathbf{K}_N$ and $\mathrm{Gr}_{\leq\lambda}$ are the fibers of $\mathbf{K}_{\widetilde{X}^\circ,N}$ and $\mathrm{Gr}_{\widetilde{X}^\circ,\leq\lambda}$ over $1 \in \widetilde{X}^\circ$. Therefore, we have a well-defined map

$$\overline{\mathrm{ev}}_{\leq\lambda} : \mathrm{Hk}_{\leq\lambda} \to [\mathbf{K}_N\backslash\mathrm{Gr}_{\leq\lambda}]. \tag{3.4}$$

For every $V \in \mathrm{Rep}(\widehat{G})$, the geometric Satake correspondence [24, Theorem 14.1] gives an $L^+\mathbb{G}$-equivariant perverse sheaf IC_V (normalized to be pure of weight zero) on the affine Grassmannian Gr. Choose a dominant coweight λ such that IC_V is supported on $\mathrm{Gr}_{\leq\lambda}$, then IC_V descends to a shifted perverse sheaf on the quotient $[\mathbf{K}_N\backslash\mathrm{Gr}_{\leq\lambda}]$ for large N. We define

$$\mathrm{IC}_V^{\mathrm{Hk}} := \overline{\mathrm{ev}}^*_{\leq\lambda}\mathrm{IC}_V \tag{3.5}$$

viewed as a complex on Hk supported on $\mathrm{Hk}_{\leq\lambda}$. Different choices of λ and N give the same $\mathrm{IC}_V^{\mathrm{Hk}}$.

3.6 Geometric Hecke operators

Let S be a scheme of finite type over k. Consider the diagram

$$\begin{tikzcd} & S \times \mathrm{Hk} \arrow[dl, "\mathrm{id}_S \times \overleftarrow{h}"'] \arrow[dr, "\mathrm{id}_S \times \overrightarrow{h}"] \arrow[rr, "\pi"] & & \widetilde{X}^\circ \\ S \times \mathrm{Bun} & & S \times \mathrm{Bun} & \end{tikzcd} \tag{3.6}$$

where $\overleftarrow{h}$ and $\overrightarrow{h}$ send $(x, \mathcal{E}, \mathcal{E}', \tau)$ to $\mathcal{E}$ and $\mathcal{E}'$, respectively. For each $V \in \mathrm{Rep}(\widehat{G})$, we define the geometric Hecke operator (relative to the base S) to be the functor

$$\begin{aligned} T_S^V &: D_c^b(S \times \mathrm{Bun}) \to D_c^b(\widetilde{X}^\circ \times S \times \mathrm{Bun}) \\ A &\mapsto (\pi \times \mathrm{id}_S \times \overrightarrow{h})_! \left((\mathrm{id}_S \times \overleftarrow{h})^* A \otimes \mathrm{IC}_V^{\mathrm{Hk}} \right). \end{aligned} \tag{3.7}$$

Definition 3.7 An *S-family of Hecke eigensheaves* is the data $(A, E_{^LG}, \{\iota_V\}_{V \in \mathrm{Rep}(\widehat{G})})$ where

- $A \in D^b(S \times \mathrm{Bun})$;
- $E_{^LG}$ is a ^LG-local system on $X^\circ \times S$. We denote the pullback of $E_{^LG}$ to $\widetilde{X}^\circ \times S$ by $E_{\widehat{G}}$, which is a $\widehat{G}$-local system with a μ_e-equivariant structure (with μ_e acting both on $\widehat{G}$ via pinned automorphisms and on $\widetilde{X}^\circ$).
- For each $V \in \mathrm{Rep}(\widehat{G})$, ι_V is an isomorphism over $\widetilde{X}^\circ \times S \times \mathrm{Bun}$

$$\iota_V : T_S^V(A) \cong E_{\widehat{G}}^V \otimes_S A,$$

  where $E_{\widehat{G}}^V$ is the local system associated with $E_{\widehat{G}}$ and the representation V of $\widehat{G}$. Here $E_{\widehat{G}}^V \otimes_S A$ means the pullback of $E_{\widehat{G}}^V \boxtimes A$ along the diagonal map $\mathrm{id}_{\widetilde{X}} \times \Delta_S \times \mathrm{id}_{\mathrm{Bun}} : \widetilde{X}^\circ \times S \times \mathrm{Bun} \to \widetilde{X}^\circ \times S \times S \times \mathrm{Bun}$.

such that the following conditions are satisfied.

(1) The isomorphisms ι_V is compatible with the tensor structure of $\mathrm{Rep}(\widehat{G})$. For details, see [5, p. 163, 164].

(2) The isomorphisms ι_V is compatible with the actions of μ_e. In other words, for $V \in \mathrm{Rep}(\widehat{G})$, let V^σ be the same vector space on which $\widehat{G}$ acts via $\widehat{G} \xrightarrow{\sigma} \widehat{G} \to \mathrm{GL}(V)$. Let $\sigma_{\widetilde{X}^\circ} : \widetilde{X}^\circ \to \widetilde{X}^\circ$ be the action of μ_e on $\widetilde{X}^\circ$. Then we have a canonical isomorphism of functors $T_S^{V^\sigma}(-) \cong (\sigma_{\widetilde{X}^\circ} \times \mathrm{id}_{S \times \mathrm{Bun}})^* T_S^V(-)$ induced from the μ_e-equivariance of the Hecke correspondence, and an isomorphism of local systems $E_{\widehat{G}}^{V^\sigma} \cong (\sigma_{\widetilde{X}^\circ} \times \mathrm{id}_S)^* E_{\widehat{G}}^V$ since $E_{\widehat{G}}$ comes from the ^LG-local system $E_{^LG}$. Then the following diagram is required to be commutative

$$\begin{tikzcd} T_S^{V^\sigma}(A) \arrow[r, "\iota_{V^\sigma}"] \arrow[d, "\wr"] & E_{\widehat{G}}^{V^\sigma} \otimes_S A \arrow[d, "\wr"] \\ (\sigma_{\widetilde{X}^\circ} \times \mathrm{id}_{S \times \mathrm{Bun}})^* T_S^V(A) \arrow[r, "(\sigma_{\widetilde{X}^\circ} \times \mathrm{id}_S)^* \iota_V"] & (\sigma_{\widetilde{X}^\circ} \times \mathrm{id}_S)^* E_{\widehat{G}}^V \otimes_S A \end{tikzcd}$$

For details, see [11, §2.4].

Theorem 3.8 *Let S be a scheme of finite type over $V_{\mathbf{P}}^{*,\mathrm{st}}$. The complex $A(\mathcal{K}, S)$ can be given the structure of an S-family of Hecke eigensheaves. We denote the corresponding eigen local system by $\mathrm{Kl}_{^LG,\mathbf{P}}(\mathcal{K}, S)$, which is a ^LG-local system over $X^\circ \times S$.*

Proof We abbreviate $A(\mathcal{K}, S)$ by A. We first prove that $T_S^V(A) \cong E_S^V \otimes_S A$ for some *complex* E_S^V on $X^\circ \times S$. Note that the Hecke operators preserve the $(\widetilde{L}_{\mathbf{P}}, \mathcal{K})$ and $(V_{\mathbf{P}} \times S, \mathrm{AS}_S)$-equivariant structures; hence, they give functors

$$T_S^V : \mathcal{D}(\mathcal{K}, S) \to \mathcal{D}(\mathcal{K}, \widetilde{X}^\circ \times S).$$

Applying Lemma 3.4(2) to $S' = \widetilde{X}^\circ \times S$, objects on the right side above are of the form $(\mathrm{id}_{\widetilde{X}^\circ} \times S \times j)_!(\mathcal{K} \boxtimes \mathrm{AS}_S \otimes \mathrm{pr}_S^* E_S^V) \cong E_S^V \otimes_S A$ for some $E_S^V \in D_c^b(\widetilde{X}^\circ \times S)$.

By the construction of the Hecke operators and its compatibility with the geometric Satake correspondence (see [5, Prop 2.8 and §2.9]), the assignment $V \mapsto E_S^V$ carries associativity and commutativity constraints for tensor functors (with respect to the sheaf-theoretic tensor product of E_S^V and $E_S^{V'}$).

It remains to show that E_S^V is a local system. We first consider the case where S is a geometric point. The argument in this case is the same as in the Iwahori level case treated in [11, §4.1, 4.2]: one first argues that $E_S^V[1]$ is a perverse sheaf and then uses the tensor structure of $V \mapsto E_S^V$ to show that it is indeed a local system. Let us only mention one key point in proving that $E^V[1]$ is perverse, that is to show an analog of [11, Remark 4.2]: The map $\pi \times \overrightarrow{h} : \overleftarrow{h}^{-1}(U) \subset \mathrm{Hk} \to \widetilde{X}^\circ \times \mathrm{Bun}$ is affine. For this we only need to argue that for any $x \in \widetilde{X}^\circ(R)$ where R is a finitely generated k-algebra, the preimage of U in the affine Grassmannian Gr_x over R is affine. By [29, Lemma 3.1], U is the nonvanishing locus of some line bundle $\mathcal{L}$ on Bun. The pullback of $\mathcal{L}$ to Gr_x has to be ample relative to the base $\mathrm{Spec}(R)$ because the relative Picard group of Gr_x is $\mathbb{Z}$ by [6, Corollary 12]. Therefore, the preimage of U in Gr_x is affine.

We then treat the general case. Since the Hecke operators T_S^V commute with pullback along a base change map $S' \to S$, so does the formation of E_S^V. Therefore, for every geometric point $s \in S$, $E_S^V|_{\widetilde{X}^\circ \times \{s\}}$ is isomorphic to E_s^V, which is a local system on $\widetilde{X}^\circ \times \{s\}$ by the argument of the previous paragraph. Let $t \in S$ be a geometric point that specializes to another geometric point s, then we have the specialization map $\mathrm{sp}_{s\to t}^V : E_s^V \to E_t^V$. Note that E_s^V and E_t^V are plain vector spaces of dimension $\dim V$. As $V \in \mathrm{Rep}(\widehat{G})$ varies, the functors $V \mapsto E_s^V$ and $V \mapsto E_t^V$ are fiber functors of the rigid tensor category $\mathrm{Rep}(\widehat{G})$ and the maps $\{\mathrm{sp}_{s\to t}^V\}_V$ give a morphism of tensor functors. Therefore, each $\mathrm{sp}_{s\to t}^V$ is an isomorphism by [3, Proposition 1.13]. This being true for every pair of specialization (t, s), E_S^V is a local system over S. The proof is complete. □

Let $\mathbb{G}_m^{\mathrm{rot}}$ be the one-dimensional torus acting simply transitively on $\widetilde{X}^\circ$. It also acts on X° via the eth power. It also acts on every standard parahoric subgroup of the loop groups L_0G and $L_\infty G$, hence on $L_{\mathbf{P}}$ and $V_{\mathbf{P}}$. We denote the actions of $\lambda \in \mathbb{G}_m^{\mathrm{rot}}$ mentioned above by $\lambda \cdot_{\mathrm{rot}} (-)$.

Viewing $L_{\mathbf{P}}$ as the Levi factor of $\mathbf{P}_\infty$, there is an action of $L_{\mathbf{P}} \rtimes \mathbb{G}_m^{\mathrm{rot}}$ on Bun where $L_{\mathbf{P}}$ changes the $\mathbf{P}_\infty^{++}$-level structures. When $S = V_{\mathbf{P}}^{*,\mathrm{st}}$, this action can be extended to every space in the Hecke correspondence diagram (3.6) by making $L_{\mathbf{P}} \rtimes \mathbb{G}_m^{\mathrm{rot}}$ act on $S = V_{\mathbf{P}}^{*,\mathrm{st}}$ in the natural way, such that all maps in (3.6) are $L_{\mathbf{P}} \rtimes \mathbb{G}_m^{\mathrm{rot}}$-equivariant. Consequently, we have

Lemma 3.9 *The ${}^LG$-local system $\widetilde{\mathrm{Kl}}_{{}^LG,m}(\mathcal{K}, V_{\mathbf{P}}^{*,\mathrm{st}})$ over $X^\circ \times V_{\mathbf{P}}^{*,\mathrm{st}}$ is equivariant under the action of $L_{\mathbf{P}} \rtimes \mathbb{G}_m^{\mathrm{rot}}$ on $X^\circ \times V_{\mathbf{P}}^{*,\mathrm{st}}$. Here $\mathbb{G}_m^{\mathrm{rot}}$ acts diagonally on $X^\circ \times V_{\mathbf{P}}^{*,\mathrm{st}}$. Therefore, $\mathrm{Kl}_{{}^LG,\mathbf{P}}(\mathcal{K}, V_{\mathbf{P}}^{*,\mathrm{st}})$ descends to an $L_{\mathbf{P}}$-equivariant $\widehat{G}$-local system over $V_{\mathbf{P}}^{*,\mathrm{st}}$, equipped with a μ_e-equivariant structure (which acts on both $\widehat{G}$ and on $V_{\mathbf{P}}^{*,\mathrm{st}}$ via the embedding $\mu_e \hookrightarrow \mathbb{G}_m^{\mathrm{rot}}$). We denote this $\widehat{G}$-local system on $V_{\mathbf{P}}^{*,\mathrm{st}}$ by $\mathrm{Kl}_{\widehat{G},\mathbf{P}}(\mathcal{K})$.*

For a stable linear functional $\phi : V_{\mathbf{P}} \to k$, viewed as a point $\phi \in V_{\mathbf{P}}^{*,\mathrm{st}}(k)$, the notation $\mathrm{Kl}_{\widehat{G},\mathbf{P}}(\mathcal{K},\phi)$ and $\mathrm{Kl}_{{}^L G,\mathbf{P}}(\mathcal{K},\phi)$ is defined as in Theorem 3.8 by taking S to be the $\operatorname{Spec} k \xrightarrow{\phi} V_{\mathbf{P}}^{*,\mathrm{st}}$.

A direct consequence of Theorem 3.8 and Lemma 3.9 is

Corollary 3.10 *For a stable linear functional $\phi : V_{\mathbf{P}} \to k$, the ${}^L G$-local system $\mathrm{Kl}_{{}^L G,\mathbf{P}}(\mathcal{K},\phi)$ over X° is the global Langlands parameter attached to the automorphic representation $\pi(\mathcal{K},\phi)$ in* Proposition 2.11. *Moreover, $\mathrm{Kl}_{\widehat{G},\mathbf{P}}(\mathcal{K},\phi)$ is isomorphic to the pullback of $\mathrm{Kl}_{\widehat{G},\mathbf{P}}(\mathcal{K})$ along the map $a_\phi : \widetilde{X}^\circ \cong \mathbb{G}_m^{\mathrm{rot}} \to V_{\mathbf{P}}^{*,\mathrm{st}}$ given by $\lambda \mapsto \lambda \cdot_{\mathrm{rot}} \phi$.*

The case considered in [11] is $\mathbf{P} = \mathbf{I}$. In this case, $L_{\mathbf{I}} = \mathbb{S}$ and $V_{\mathbf{I}} = \oplus_{i=0}^r V_{\mathbf{I}}(\alpha_i)$ for affine simple roots $\{\alpha_i\}$ of G. If G is adjoint, we may identify the quotient $[V_{\mathbf{I}}^{*,\mathrm{st}}/\mathbb{S}]$ with $\mathbb{G}_m$ (via projection to the factor $V_{\mathbf{P}}(\alpha_0)^*$). So in this case the Kloosterman sheaf $\mathrm{Kl}_{\widehat{G},\mathbf{I}}(\mathcal{K})$ descends to a $\widehat{G}$-local system on $\mathbb{G}_m$, without any dependence on the functional ϕ.

3.11 Calculation of the local system

The moduli stack $\mathrm{Bun}_G(\widetilde{\mathbf{P}}_0, \mathbf{P}_\infty^+)$ has a unique point $\mathcal{E}$ which has trivial automorphism group. Let $\mathfrak{G}$ be the automorphism group of $\mathcal{E}|_{X-\{1\}}$ preserving the level structures at 0 and ∞. This is a group ind-scheme over k. Now consider $1 \in \widetilde{X}$, a preimage of $1 \in X$. To distinguish them, we denote the $1 \in \widetilde{X}$ by $\widetilde{1}$. Note that the pullback of $\mathcal{E}$ along $\widetilde{X}^\circ \to X$ is a $\mathbb{G}$-bundle over $\widetilde{X}^\circ$. Therefore, for $g \in \mathfrak{G}$, we may talk about the relative position of the action of g on the pullback $\mathcal{E}|_{\widetilde{X}}$ near $\widetilde{1}$.

Let $\lambda \in \mathbb{X}_*(\mathbb{T})$ be a dominant coweight of $\mathbb{G}$. We have a closed subscheme $\mathfrak{G}_{\le\lambda} \subset \mathfrak{G}$ consisting of those automorphisms of $\mathcal{E}|_{X-\{1\}}$ whose relative position near $\widetilde{1}$ is bounded by $\le \lambda$. For sufficiently large N, evaluating an automorphism $g \in \mathfrak{G}$ in the formal neighborhood of $\widetilde{1}$ gives a morphism

$$\mathrm{ev}_{\mathfrak{G},\le\lambda} : \mathfrak{G}_{\le\lambda} \to [\mathbf{K}_N \backslash \mathrm{Gr}_{\le\lambda}]. \tag{3.8}$$

similar to the one defined in (3.3) for the Hecke stack.

The intersection complex IC_λ of $\mathrm{Gr}_{\le\lambda}$ corresponds, under the geometric Satake equivalence, to the irreducible representation V_λ of $\widehat{G}$ with highest weight λ. Since IC_λ is $\mathbf{K}_N$-equivariant, we view IC_λ as a shifted perverse sheaf on $[\mathbf{K}_N \backslash \mathrm{Gr}_{\le\lambda}]$.

We have two more evaluation maps, at 0 and ∞

$$(\mathrm{ev}_0, \mathrm{ev}_\infty) : \mathfrak{G}_{\le\lambda} \to \widetilde{\mathbf{P}}_0 \times \mathbf{P}_\infty^+.$$

Composing with the projections $\widetilde{\mathbf{P}}_0 \to \widetilde{L}_{\mathbf{P}}$ and $\mathbf{P}_\infty^+ \to V_{\mathbf{P}}$, we get

$$(f', f'') : \mathfrak{G}_{\le\lambda} \to \widetilde{L}_{\mathbf{P}} \times V_{\mathbf{P}}. \tag{3.9}$$

Proposition 3.12 *Let* $\mathrm{Four}_\psi : D^b_c(V_{\mathbf{P}}) \to D^b_c(V^*_{\mathbf{P}})$ *be the Fourier-Deligne transform (without cohomological shift). We have*

$$\mathrm{Kl}^{V_\lambda}_{\widehat{G},\mathbf{P}}(\mathcal{K}) \cong \mathrm{Four}_\psi\left(f''_!(f'^*\mathcal{K} \otimes \mathrm{ev}^*_{\mathfrak{G},\leq\lambda}\mathrm{IC}_\lambda)\right)|_{V^{*,\mathrm{st}}_{\mathbf{P}}}.$$

Proof We shall work with the base $S = V^{*,\mathrm{st}}_{\mathbf{P}}$. By construction, $\mathrm{Kl}^{V_\lambda}_{\widehat{G},\mathbf{P}}(\mathcal{K})$ is the restriction of $T^{V_\lambda}_S(A(\mathcal{K}, S))$ to the point $\{\widetilde{1}\} \times S \times \{\star\} \subset \widetilde{X}^\circ \times S \times \mathrm{Bun}$. Let $\mathrm{Hk}_{\widetilde{1}} \subset \mathrm{Hk}$ be the preimage of $(\widetilde{1}, \star)$ under $(\pi, \overrightarrow{h})$, and $\mathrm{Hk}^U_{\widetilde{1}} \subset \mathrm{Hk}_{\widetilde{1}}$ be the preimage of U under $\overleftarrow{h}_{\widetilde{1}}$ (fiber of $\overleftarrow{h}$ over $\widetilde{1}$). By proper base change and the definition of the Hecke operators (3.7), we have

$$\mathrm{Kl}^{V_\lambda}_{\widehat{G},\mathbf{P}}(\mathcal{K}) = \mathrm{pr}_{S,!}((\mathrm{id}_S \times \overleftarrow{h}_{\widetilde{1}})^*\mathrm{AS}_S \otimes \mathrm{IC}^{\mathrm{Hk}_{\widetilde{1}}}_\lambda). \tag{3.10}$$

where

$$S \times U \xleftarrow{\mathrm{id}_S \times \overleftarrow{h}_{\widetilde{1}}} S \times \mathrm{Hk}^U_{\widetilde{1}} \xrightarrow{\mathrm{pr}_S} S$$

and $\mathrm{IC}^{\mathrm{Hk}_{\widetilde{1}}}_\lambda$ is the restriction of $\mathrm{IC}^{\mathrm{Hk}}_{V_\lambda}$ [see (3.5)] to $\mathrm{Hk}_{\widetilde{1}}$.

By definition, $\mathrm{Hk}^U_{\widetilde{1}}$ classifies isomorphisms pairs $(\mathcal{E}', \tau)$ where $\mathcal{E}' \in U$ and τ is an isomorphism between $\star|_{\widetilde{X}^\circ-\{\widetilde{1}\}}$ and $\mathcal{E}'|_{\widetilde{X}^\circ-\{\widetilde{1}\}}$ preserving the $\mathbf{P}^+_0$ and $\mathbf{P}^{++}_\infty$-level structures. Since both $\star$ and $\mathcal{E}' \in U$ map to the open point $\mathcal{E}$ in $\mathrm{Bun}_G(\widetilde{\mathbf{P}}_0, \mathbf{P}^+_\infty)$, we may define an isomorphism $\iota : \mathfrak{G} \cong \mathrm{Hk}^U_{\widetilde{1}}$ as follows. For an automorphism α of $\mathcal{E}|_{\widetilde{X}^\circ-\{\widetilde{1}\}}$, viewed as an automorphism of $\star|_{\widetilde{X}^\circ-\{\widetilde{1}\}}$ not necessarily preserving the $\mathbf{P}^+_0$ and $\mathbf{P}^{++}_\infty$-level structures, $\iota(\alpha)$ is the pair $(\mathcal{E}'_\alpha, \alpha) \in \mathrm{Hk}^U_{\widetilde{1}}$, where $\mathcal{E}'_\alpha$ is the unique point in U obtained by modifying the $\mathbf{P}^+_0$ and $\mathbf{P}^{++}_\infty$-level structures of $\star$ so that α becomes an isomorphism between $\star|_{\widetilde{X}^\circ-\{\widetilde{1}\}}$ and $\mathcal{E}'_\alpha|_{\widetilde{X}^\circ-\{\widetilde{1}\}}$ preserving the $\mathbf{P}^+_0$ and $\mathbf{P}^{++}_\infty$-level structures.

Under the isomorphism ι, $\overleftarrow{h}_{\widetilde{1}} : \mathrm{Hk}^U_{\widetilde{1}} \to U$ is identified with (f', f'') defined in (3.9). The $\mathfrak{G}_{\leq\lambda}$ corresponds to $\mathrm{Hk}^U_{\widetilde{1},\leq\lambda}$, and the map $\mathrm{ev}_{\mathfrak{G},\leq\lambda}$ in (3.8) corresponds to the fiber of $\mathrm{ev}_{\leq\lambda}$ in (3.3) over $\widetilde{1}$. Therefore, using the identification $\iota_{\leq\lambda} : \mathfrak{G}_{\leq\lambda} \cong \mathrm{Hk}^U_{\widetilde{1},\leq\lambda}$ and (3.10), we have

$$\begin{aligned}\mathrm{Kl}^{V_\lambda}_{\widehat{G},\mathbf{P}}(\mathcal{K}) &\cong \mathrm{pr}_{S,!}(f'^*\mathcal{K} \otimes (\mathrm{id}_S \times f'')^*\mathrm{AS}_S \otimes \mathrm{ev}^*_{\mathfrak{G},\leq\lambda}\mathrm{IC}_\lambda)\\ &= \mathrm{pr}_{V^*_{\mathbf{P}},!}(f'^*\mathcal{K} \otimes (\mathrm{id}_{V^*_{\mathbf{P}}} \times f'')^*\langle\cdot,\cdot\rangle^*\mathrm{AS}_\psi \otimes \mathrm{ev}^*_{\mathfrak{G},\leq\lambda}\mathrm{IC}_\lambda)|_{V^{*,\mathrm{st}}_{\mathbf{P}}}\end{aligned}$$

Using proper base change, the above is further isomorphic to

$$\begin{aligned}&\mathrm{pr}_{V^*_{\mathbf{P}},!}\left((\mathrm{id}_{V^*_{\mathbf{P}}} \times f'')_!(f'^*\mathcal{K} \otimes \mathrm{ev}^*_{\mathfrak{G},\leq\lambda}\mathrm{IC}_\lambda) \otimes \langle\cdot,\cdot\rangle^*\mathrm{AS}_\psi\right)|_{V^{*,\mathrm{st}}_{\mathbf{P}}}\\ &= \mathrm{Four}_\psi\left(f''_!(f'^*\mathcal{K} \otimes \mathrm{ev}^*_{\mathfrak{G},\leq\lambda}\mathrm{IC}_\lambda)\right)|_{V^{*,\mathrm{st}}_{\mathbf{P}}}.\end{aligned}$$

$\square$

4 Unipotent monodromy

The goal of this section is to study the monodromy of the Kloosterman sheaves $\mathrm{Kl}_{\widehat{G},\mathbf{P}}(\mathcal{K}, \phi)$ at 0, especially when $\mathcal{K}$ is trivial.

4.1 Lusztig's theory of two-sided cells

Let W_{aff} be the affine Weyl group attached to $G(K)$. There is a preorder $\leq_{\mathrm{LR}}$ on W_{aff} defined in [16, 4.2]. The partial order generated by $\leq_{\mathrm{LR}}$ gives a partition of W_{aff} into finitely many *two-sided cells* and a partial order among them. The largest cell is the singleton $\{1\}$; the smallest one contains the longest element w_0 in the Weyl group $W_{\mathbf{Q}}$ of a special parahoric subgroup $\mathbf{Q}$. By [17, 1.1,1.2], each cell $\underline{c}$ is assigned an integer $a(\underline{c})$ satisfying $0 \leq a(\underline{c}) \leq \min\{\ell(w)|w \in \underline{c}\}$.

Let $\widehat{G}^{\sigma,\circ}$ be the neutral component of the fixed point of σ on $\widehat{G}$.

Theorem 4.2 (Lusztig [19, Theorem 4.8]) *There is an order preserving bijection*

$$\{\text{two} - \text{sided cells in } W_{\mathrm{aff}}\} \longleftrightarrow \{\text{unipotent conjugacy classes in } \widehat{G}^{\sigma,\circ}\}$$

such that if $\underline{c} \leftrightarrow \underline{u}$, then $a(\underline{c}) = \dim \mathcal{B}_u$ where $\mathcal{B}_u$ is the Springer fiber of $u \in \underline{u}$ (inside the flag variety of $\widehat{G}^{\sigma,\circ}$).

Remark 4.3 Lusztig's papers [16–19] only dealt with the case of split G. However, for our quasi-split G, the affine Weyl group W_{aff} is isomorphic to the affine Weyl group of a split group G' as Coxeter groups. Moreover, one can choose G' so that $\widehat{G}' = \widehat{G}^{\sigma,\circ}$. Therefore, the quasi-split case of the above theorem follows from the split case.

Definition 4.4 For a standard parahoric subgroup $\mathbf{P} \subset G(K)$, let $\underline{c}_{\mathbf{P}}$ be the two-sided cell in W_{aff} containing the longest element $w_{\mathbf{P}} \in W_{\mathbf{P}}$ (the Weyl group of $L_{\mathbf{P}}$). We define $\underline{u}_{\mathbf{P}}$ to be the unipotent class of $\widehat{G}^{\sigma,\circ}$ corresponding to $\underline{c}_{\mathbf{P}}$ under Theorem 4.2.

The main result of this section is

Theorem 4.5 *Assume $\mathcal{K} = \mathbf{1}$ is the trivial character sheaf on $\widetilde{L}_{\mathbf{P}}$ and G is split. Then for any stable functional $\phi : V_{\mathbf{P}} \to k$, the local monodromy of the $\widehat{G}$-local system $\mathrm{Kl}_{\widehat{G},\mathbf{P}}(\mathbf{1}, \phi)$ at 0 is tame, and the monodromy action of any topological generator of the tame inertia at 0 is by an element in the conjugacy class $\underline{u}_{\mathbf{P}}$.*

The proof of the theorem relies on deep results of Lusztig and Bezrukavnikov on cells in affine Weyl groups and occupies Sects. 4.11–4.18. If $\mathbf{P} = \mathbf{I}$, then $\underline{c}_{\mathbf{P}}$ is the maximal cell; hence, $\underline{u}_{\mathbf{P}}$ is the regular unipotent class. This case of the theorem was proved in [11, Theorem 1(2)]. We expect that the analogous result also holds when G is quasi-split; see Sect. 4.10 for more general conjectures.

Before giving the proof of Theorem 4.5, we will give explicit calculation of the unipotent classes $\underline{u}_{\mathbf{P}}$. This will be completed in Sect. 4.9. To achieve that, we first collect some properties of the unipotent classes $\underline{u}_{\mathbf{P}}$. We still work in the generality where G is quasi-split as in Sect. 1.4.4.

Lemma 4.6 *For any standard parahoric* $\mathbf{P}$ *and any* $u \in \underline{u}_{\mathbf{P}}$*, we have*

$$\dim \mathcal{B}_u = \ell(w_{\mathbf{P}}).$$

Proof By definition, $\dim \mathcal{B}_u = a(\underline{c}_{\mathbf{P}})$, and therefore, it suffices to show that $a(\underline{c}_{\mathbf{P}}) = \ell(w_{\mathbf{P}})$. For this, we may assume that G is simply connected. Let $a = \ell(w_{\mathbf{P}})$. Let $\mathbf{H} := C_c(\mathbf{I}\backslash G(K)/\mathbf{I})$ be the Iwahori Hecke algebra, and let $\{C_w\}_{w \in W_{\mathrm{aff}}}$ be its Kazhdan-Lusztig basis. By definition, $a(\underline{c}_{\mathbf{P}})$ is equal to $a(w_{\mathbf{P}})$, which is the largest power of $q^{1/2}$ appearing in the coefficient of $C_{w_{\mathbf{P}}}$ in $C_x C_y$ for some $x, y \in W_{\mathrm{aff}}$. Since $C_{w_{\mathbf{P}}}^2 = f(q^{1/2}) C_{w_{\mathbf{P}}}$ for $f(q^{1/2}) = q^{-a/2} \sum_{x \in W_{\mathbf{P}}} q^{\ell(x)}$ whose highest degree term is $q^{a/2}$, we have $a(w_{\mathbf{P}}) \geq a$. On the other hand, one always has $a(w_{\mathbf{P}}) \leq \ell(w_{\mathbf{P}}) = a$ by [21, II, Proposition1.2], and therefore, $a(\underline{c}_{\mathbf{P}}) = a(w_{\mathbf{P}}) = a$. $\square$

4.7 The class $\underline{u}_{\mathbf{P}}$ via truncated induction

We now give an alternative description of the unipotent class $\underline{u}_{\mathbf{P}}$. To state it, we recall the truncated induction defined by Lusztig [15] in a special case, which was first constructed by Macdonald [23]. Let $W = \mathbb{W}^{\sigma}$ be the F-Weyl group of G (this is also $W_{\mathbf{Q}}$ for a special parahoric $\mathbf{Q}$ and is also the Weyl group of $\widehat{G}^{\sigma,\circ}$), then $\mathfrak{a} = \mathbb{X}_*(\mathbb{T})^{\sigma}_{\mathbb{C}}$ is the reflection representation of W. The sign representation ϵ of $W_{\mathbf{P}}$ appears in $\mathrm{Res}^{W}_{W_{\mathbf{P}}} \mathrm{Sym}^{\ell(w_{\mathbf{P}})}(\mathfrak{a})$ with multiplicity one. The truncated induction $j^{W}_{W_{\mathbf{P}}}(\epsilon)$ is the unique irreducible W-submodule of $\mathrm{Sym}^{\ell(w_{\mathbf{P}})}(\mathfrak{a})$ that contains the sign representation of $W_{\mathbf{P}}$.

Proposition 4.8 *Let* $\mathbf{P}$ *be a standard parahoric subgroup of* $G(K)$*.*

(1) Under the Springer correspondence, $j^{W}_{W_{\mathbf{P}}}(\epsilon)$ *corresponds to the unipotent class* $\underline{u}_{\mathbf{P}}$ *of* $\widehat{G}^{\sigma,\circ}$ *together with the trivial local system on it.*

(2) Suppose $\mathbf{P} \subset \mathbf{Q}$ *for some special parahoric* $\mathbf{Q}$*, then* $\mathbf{P}$ *corresponds to a standard parabolic subgroup* $P \subset L_{\mathbf{Q}}$*, hence also to a standard parabolic subgroup* $\widehat{P} \subset \widehat{G}^{\sigma,\circ}$*. Then* $\underline{u}_{\mathbf{P}}$ *is the Richardson class attached to* $\widehat{P}$ *(i.e., the unipotent class in* $\widehat{G}^{\sigma,\circ}$ *that contains a dense open subset of the unipotent radical of* $\widehat{P}$*).*

Proof Remark 4.3 allows us to reduce to the case G is split.

(1) The representation $j^{W}_{W_{\mathbf{P}}}(\epsilon)$ is obtained from truncated induction from a special representation of the parahoric subgroup $W_{\mathbf{P}}$; therefore, it belongs to the set $\overline{\mathcal{S}}_W$ defined in [22, §1.3]. By [22, Theorem 1.5(a)], under the Springer correspondence, $\overline{\mathcal{S}}_W$ is exactly the set of irreducible representations of W that correspond to trivial local systems on unipotent orbits. Let $\underline{u}$ (together with the trivial local system on it) be the unipotent class in $\widehat{G}$ that corresponds to $j^{W}_{W_{\mathbf{P}}}(\epsilon)$. By [22, Theorem 1.5(b1)], the dimension of the Springer fiber $\mathcal{B}_u$ for any $u \in \underline{u}$ is the lowest degree b for which ϵ appears in $\mathrm{Sym}^b(\mathfrak{a})$ as a $W_{\mathbf{P}}$-submodule. Therefore, $\dim \mathcal{B}_u = \ell(w_{\mathbf{P}})$ for any $u \in \underline{u}$.

Recall that $\mathbf{H} := C_c(\mathbf{I}\backslash G(K)/\mathbf{I})$ is the Iwahori Hecke algebra, and let $\mathbf{H}_{\mathbf{P}} := C(\mathbf{I}\backslash \mathbf{P}/\mathbf{I})$ be its parahoric subalgebra. Let M be an irreducible $\mathbf{H}$-module such that $M^{\mathbf{H}_{\mathbf{P}}} \neq 0$ and that C_w acts on M by zero for any w lying in cells $\underline{c} \leq \underline{c}_{\mathbf{P}}$. Such an $\mathbf{H}$-module can be found in the cell subquotient $\mathbf{H}_{\underline{c}_{\mathbf{P}}}$. The Langlands parameter of M is a triple (u', s, ρ) where u' is a unipotent class in $\widehat{G}$. By the construction of Lusztig's bijection Theorem 4.2, $u' \in \underline{u}_{\mathbf{P}}$. Consider the $\mathbf{H}$-module ${}^*M$ (changing the action of T_s to $-qT_s^{-1}$ for simple reflections $s \in W_{\mathrm{aff}}$, see [19, §1.8]), then ${}^*M$ contains a copy of the sign representation of $\mathbf{H}_{\mathbf{P}}$. Applying [19, Lemma 7.2] to ${}^*M$, we conclude that u' lies in the closure of $\underline{u}$. Since we also have $\dim \mathcal{B}_{u'} = a(\underline{c}_{\mathbf{P}}) = \ell(w_{\mathbf{P}})$ by Theorem 4.2 and Lemma 4.6, we must have $\underline{u} = \underline{u}_{\mathbf{P}}$.

(2) follows from [14, Theorem 3.5]. □

4.9 Tables of unipotent monodromy

We shall assume $\mathrm{char}(k)$ is large so that $\mathbf{P}$ corresponds to a unique regular elliptic number $m = m(\mathbf{P})$ of $(\mathbb{W}, \sigma)$. We denote $\underline{u}_{\mathbf{P}}$ by $\underline{u}_m$. In this subsection, we list $\underline{u}_{\mathbf{P}}$ for all admissible $\mathbf{P}$ and quasi-split G.

When G is of classical type, Proposition 4.8 allows us to determine $\underline{u}_{\mathbf{P}}$. The classification of admissible parahorics will be reviewed in Sects. 6–8. We denote the unipotent classes $\underline{u}_{\mathbf{P}}$ by the sizes of their Jordan blocks: The notation $\boxed{i}^j$ denotes j Jordan blocks each of size i.

G	m	$\underline{u}_m$	$\widehat{G}^\sigma$
A_{n-1}	n	$\boxed{n}$	A_{n-1}
${}^2A_{n-1}$	$m = \frac{2n}{d};\ d \mid n$	$\boxed{\frac{m}{2}}^d$	$B_{\frac{n-1}{2}}$
n odd	$m = \frac{2(n-1)}{d} > 2;\ \frac{n-1}{d}$ odd	$\boxed{\frac{m}{2}}^d \boxed{1}$	
${}^2A_{n-1}$	$m = \frac{2n}{d};\ \frac{n}{d}$ odd	$\boxed{\frac{m}{2}}^{d-2} \boxed{\frac{m}{2}+1} \boxed{\frac{m}{2}-1}$	$C_{\frac{n}{2}}$
n even	$m = \frac{2(n-1)}{d} > 2;\ d \mid n-1$	$\boxed{\frac{m}{2}}^{d-1} \boxed{\frac{m}{2}+1}$	
B_n	$m = 2n/d;\ d \mid n$	$\boxed{m}^d$	C_n
C_n	$m = 2n/d;\ d \mid n, d$ odd	$\boxed{m}^{d-1} \boxed{m+1}$	B_n
	$m = 2n/d;\ d \mid n, d$ even	$\boxed{m}^{d-2} \boxed{m-1} \boxed{m+1} \boxed{1}$	
D_n	$m = 2n/d;\ d \mid n, d$ even	$\boxed{m}^{d-2} \boxed{m+1} \boxed{m-1}$	D_n
	$m = \frac{2(n-1)}{d};\ d \mid n-1, d$ odd	$\boxed{m}^{d-1} \boxed{m+1} \boxed{1}$	
${}^2D_n$	$m = 2n/d;\ d \mid n, d$ odd	$\boxed{m}^{d-1} \boxed{m-1}$	B_{n-1}
	$m = \frac{2(n-1)}{d};\ d \mid n-1, d$ even	$\boxed{m}^d \boxed{1}$	

For exceptional groups, we use the tables in [9, §7.1] to list all regular elliptic numbers and use Bala-Carter's notation [2, §13.1] to denote unipotent classes. In many cases, the fact that $\dim \mathcal{B}_u = \ell(w_{\mathbf{P}})$ from Lemma 4.6 is already enough to determine the class $\underline{u}_{\mathbf{P}}$. In the remaining cases, G is always split, and $\underline{u}_{\mathbf{P}}$ is determined by the following methods.

- When $\mathbf{P} \subset \mathbb{G}(\mathcal{O}_K)$, we use Proposition 4.8 to conclude that $\underline{u}_{\mathbf{P}}$ is the S-distinguished unipotent classes assigned to these admissible parahoric subgroups in [9, §7.3]. The weighted Dynkin diagram of the distinguished unipotent class $\underline{u}_{\mathbf{P}}$ is obtained by putting 0 on simple roots in $L_{\mathbf{P}}$ and putting 2 elsewhere.
- For those $\mathbf{P}$ not contained in $\mathbb{G}(\mathcal{O}_K)$, we first use the tables in [22, §7.1] to find the dimension of the truncated induction $j_{W_{\mathbf{P}}}^{W}(\epsilon)$ and then use the tables in [2, §13.3] to find the corresponding unipotent classes.

G	m	$\underline{u}_m$	$\widehat{G}^\sigma$	G	m	$\underline{u}_m$	$\widehat{G}^\sigma$
E_6	3	$2A_2+A_1$	E_6	E_8	2	$4A_1$	E_8
	6	$E_6(a_3)$			3	$2A_2+2A_1$	
	9	$E_6(a_1)$			4	$2A_3$	
	12	E_6			5	A_4+A_3	
E_7	2	$4A_1$	E_7		6	$E_8(a_7)$	
	6	$E_7(a_5)$			8	A_7	
	14	$E_7(a_1)$			10	$E_8(a_6)$	
	18	E_7			12	$E_8(a_5)$	
F_4	2	$A_1+\widetilde{A}_1$	F_4		15	$E_8(a_4)$	
	3	$\widetilde{A}_2+A_1$			20	$E_8(a_2)$	
	4	$F_4(a_3)$			24	$E_8(a_1)$	
	6	$F_4(a_2)$			30	E_8	
	8	$F_4(a_1)$		G_2	2	$\widetilde{A}_1$	G_2
	12	F_4			3	$G_2(a_1)$	
${}^2E_6$	2	A_1			6	G_2	
	4	$A_2+\widetilde{A}_1$		${}^3D_4$	3	A_1	
	6	$F_4(a_3)$			6	$G_2(a_1)$	
	12	$F_4(a_1)$			12	G_2	
	18	F_4					

4.10 Conjectural description of the tame monodromy for general $\mathcal{K}$

Finally, we give a conjectural description of the local monodromy of $\mathrm{Kl}_{{}^LG,\mathbf{P}}(\mathcal{K},\phi)$ at 0 for quasi-split G and general character sheaves $\mathcal{K}$ on $\widetilde{L}_{\mathbf{P}}$. By the sheaf-to-function correspondence, $\mathcal{K}$ gives a character $\chi : \widetilde{L}_{\mathbf{P}}(k) \to \overline{\mathbb{Q}}_\ell^\times$. First of all $\mathrm{Kl}_{{}^LG,\mathbf{P}}(\mathcal{K},\phi)$ is tame at 0. Recall that the tame inertia $\mathcal{I}_0^t \cong \varprojlim_{(n,p)=1} \mu_n(\bar{k})$ and in particular $\mathcal{I}_0^t \twoheadrightarrow \mu_e(k)$. Let $\xi \in \mathcal{I}_0^t$ be a generator that maps to $\sigma \in \mu_e$. We predict that under the monodromy representation $\mathcal{I}_0^t \to {}^LG$ associated with $\mathrm{Kl}_{{}^LG,\mathbf{P}}(\mathcal{K},\phi)$, ξ should go to $(\kappa u, \sigma) \in \widehat{G}^\sigma \times \sigma \subset {}^LG$ where κu is the Jordan decomposition into the semisimple part and unipotent part of an element in $\widehat{G}^\sigma$.

We first give a conjectural description of the semisimple part κ. Up to $\widehat{G}$-conjugacy, we may arrange that $\kappa \in \widehat{T}$. By further $\widehat{T}$-conjugation, only the image of κ in the σ-coinvariants $\widehat{T}_\sigma$ is well defined. The character $k^\times \otimes \mathbb{X}_*(\mathbb{T})^\sigma = \mathbb{S}(k) \to L_{\mathbf{P}} \xrightarrow{\chi} \overline{\mathbb{Q}}_\ell^\times$ gives a homomorphism $\chi' : k^\times \to \widehat{T}_\sigma = \mathrm{Hom}(\mathbb{X}_*(\mathbb{T})^\sigma, \overline{\mathbb{Q}}_\ell^\times)$. Local class field theory identifies $k^\times$ with the tame inertia of $\mathrm{Gal}(F_0^{\mathrm{ab}}/F_0)$ (where F_0^{ab} is a maximal abelian extension of F_0); hence, there is a canonical surjection $\mathcal{I}_0^t \twoheadrightarrow k^\times$. Composing with χ' we get a homomorphism $\chi'' : \mathcal{I}_0^t \to \widehat{T}_\sigma$. The image of κ in $\widehat{T}_\sigma$ should be $\chi''(\xi)$.

Next we give a conjectural description of the unipotent part u. Let $W_{\kappa,\mathrm{aff}} \subset W_{\mathrm{aff}}$ be the subgroup generated by affine reflections whose vector part $\alpha \in \Phi$ is such that the composition $\mathbb{G}_m(k) \xrightarrow{\alpha^\vee} \mathbb{S}(k) \to L_{\mathbf{P}}(k) \xrightarrow{\chi} \overline{\mathbb{Q}}_\ell^\times$ is trivial. It can be shown that $W_{\kappa,\mathrm{aff}}$ is an affine Coxeter group. In fact $W_{\kappa,\mathrm{aff}}$ can be identified with the affine Weyl group of the group dual to $\widehat{G}_\kappa^{\sigma,\circ}$, the centralizer in $\widehat{G}^{\sigma,\circ}$ of an element $\widetilde{\kappa} \in \widehat{T}^{\sigma,\circ}$ lifting $\kappa \in \widehat{T}_\sigma$. The longest element $w_{\mathbf{P}} \in W_{\mathbf{P}}$ belongs to $W_{\kappa,\mathrm{aff}}$, hence lies in a two-sided cell $\underline{c}_{\kappa,\mathbf{P}}$ of $W_{\kappa,\mathrm{aff}}$. By Lusztig's Theorem 4.2, $\underline{c}_{\kappa,\mathbf{P}}$ corresponds to a unipotent class $\underline{u}_{\kappa,\mathbf{P}}$ of $\widehat{G}_\kappa^{\sigma,\circ}$. The unipotent element u should lie in $\underline{u}_{\kappa,\mathbf{P}}$.

4.11 Hecke operators at 0

The rest of this section is devoted to the proof of Theorem 4.5. We first set up some notation. Fix a stable functional $\phi : V_{\mathbf{P}} \to k$. Since $\mathcal{K} = \mathbf{1}$ is the trivial local system, we simply denote the Kloosterman sheaf $\mathrm{Kl}_{\widehat{G},\mathbf{P}}(\mathcal{K},\phi)$ by $\mathrm{Kl}_{\widehat{G},\mathbf{P}}(\phi)$. Comparing a general split G with its simply connected cover G^{sc}, we have a canonical homomorphism $\nu : \widehat{G} \to \widehat{G^{\mathrm{sc}}} = \widehat{G}^{\mathrm{ad}}$. Our construction of the Kloosterman local system is compatible with the homomorphism ν in the sense that the local system $\mathrm{Kl}_{\widehat{G},\mathbf{P}}(\phi)$ induces the local system $\mathrm{Kl}_{\widehat{G^{\mathrm{sc}}},m}(\phi)$ via ν. Therefore, it suffices to prove Theorem 4.5 for simply connected G.

In the sequel, we shall base change all spaces to $\bar{k}$ without changing notation. When S is a point ϕ : Spec $\bar{k} \to V_{\mathbf{P}}^{*,\mathrm{st}}$, we denote the category of automorphic sheaves $\mathcal{D}(\mathbf{1},S)$ by $\mathcal{D}(\phi)$. Since $\mathcal{K}$ is trivial, objects in $\mathcal{D}(\phi)$ descends to $\mathrm{Bun}_G(\mathbf{P}_0,\mathbf{P}_\infty^{++})$. Therefore, we may identify $\mathcal{D}(\phi)$ with a full subcategory of $D_c^b(\mathrm{Bun}(\mathbf{P}_0,\mathbf{P}_\infty^{++}))$. Let $A(\phi) \in \mathcal{D}(\phi)$ be the automorphic sheaf $A(\mathbf{1},S)$ in Lemma 3.4(1), which is a Hecke eigensheaf with eigen local system $\mathrm{Kl}_{\widehat{G},\mathbf{P}}(\phi)$ by Theorem 3.8 .

The Hecke correspondence at 0 is defined as the moduli stack of $(\mathcal{E},\mathcal{E}',\tau)$ where $\mathcal{E},\mathcal{E}' \in \mathrm{Bun}_G(\mathbf{P}_0,\mathbf{P}_\infty^{++})$ and $\tau : \mathcal{E}|_{X-\{0\}} \xrightarrow{\sim} \mathcal{E}'|_{X-\{0\}}$. We have a diagram similar to (3.6)

$$\begin{tikzcd} & \mathrm{Hk}_0 \arrow[dl, "\overleftarrow{h}_0"'] \arrow[dr, "\overrightarrow{h}_0"] & \\ \mathrm{Bun}_G(\mathbf{P}_0,\mathbf{P}_\infty^{++}) & & \mathrm{Bun}_G(\mathbf{P}_0,\mathbf{P}_\infty^{++}) \end{tikzcd} \tag{4.1}$$

Let $\mathrm{Fl}_{\mathbf{P}_0} = L_0G/\mathbf{P}_0$ be the affine partial flag variety. Consider the category $D_{\mathbf{P}_0}^b(\mathrm{Fl}_{\mathbf{P}_0})$ of $\mathbf{P}_0$-equivariant constructible complexes on $\mathrm{Fl}_{\mathbf{P}_0}$ with finite type support. This category has a monoidal structure given by convolution of sheaves $(C_1,C_2) \mapsto C_1 \overset{\mathbf{P}_0}{*} C_2$ with unit object $\delta_{\mathbf{P}_0}$ given by the constant sheaf supported on the identity double coset. The set of $\mathbf{P}_0$-orbits on $\mathrm{Fl}_{\mathbf{P}_0}$ is isomorphic to the double quotient $W_{\mathbf{P}_0}\backslash\widetilde{W}/W_{\mathbf{P}_0}$ as a poset. For any $[w] \in W_{\mathbf{P}_0}\backslash\widetilde{W}/W_{\mathbf{P}_0}$, we denote the corresponding orbit closure by $\mathrm{Fl}_{\mathbf{P}_0,\le[w]}$. We also have a substack $\mathrm{Hk}_{0,\le[w]} \subset \mathrm{Hk}_0$ by requiring that the relative position of τ at 0 be bounded by the double coset $[w]$. Similar to the map $\mathrm{ev}_{\le\lambda}$ defined for $\mathrm{Hk}_{\le\lambda}$ in (3.3), there is a morphism

$$\mathrm{ev}_{0,\le[w]}:\mathrm{Hk}_{0,\le[w]}\to[\mathbf{K}_{[w]}\backslash\mathrm{Fl}_{\mathbf{P}_0,\le[w]}]$$

recording the relative position of the modification τ at 0. Here $\mathbf{K}_{[w]}$ is any finite-dimensional quotient of $\mathbf{P}_0$ through which the left action of $\mathbf{P}_0$ on $\mathrm{Fl}_{\mathbf{P}_0,\le[w]}$ factors. For any object $C\in D^b_{\mathbf{P}_0}(\mathrm{Fl}_{\mathbf{P}_0})$ supported on $\mathrm{Fl}_{\mathbf{P}_0,\le[w]}$, we define

$$C^{\mathrm{Hk}_0}:=\mathrm{ev}^*_{0,\le[w]}C$$

viewed as a complex on Hk_0. Again C^{Hk_0} is independent of the choices of $[w]$ and $\mathbf{K}_{[w]}$.

We define the geometric Hecke functor at 0 to be

$$\begin{aligned}T_0(-,-):D^b_{\mathbf{P}_0}(\mathrm{Fl}_{\mathbf{P}_0})\times\mathcal{D}(\phi)&\to\mathcal{D}(\phi)\\ (C,A)&\mapsto\overrightarrow{h}_{0,!}(\overleftarrow{h}^*_0A\otimes C^{\mathrm{Hk}_0}).\end{aligned}$$

The monoidal structure on $D^b_{\mathbf{P}_0}(\mathrm{Fl}_{\mathbf{P}_0})$ is compatible with the composition of geometric Hecke operators: There is natural isomorphism of endo-functors on $\mathcal{D}(\phi)$

$$T_0(C_1,T_0(C_2,-))\cong T_0(C_1\overset{\mathbf{P}_0}{*}C_2,-)$$

which satisfies associativity. By Lemma 3.4(1), $\mathcal{D}(\phi)\cong D^b_c(\mathrm{Vect})$ with the inverse functor given by $(-)\otimes A(\phi)$; therefore, we have $T_0(C,A(\phi))\cong e(C)\otimes A(\phi)$ for a well-defined monoidal functor

$$e:D^b_{\mathbf{P}_0}(\mathrm{Fl}_{\mathbf{P}_0})\to D^b_c(\mathrm{Vect}).$$

We equip $D^b_{\mathbf{P}_0}(\mathrm{Fl}_{\mathbf{P}_0})$ with the perverse t-structure by viewing it as a full subcategory of complexes on $\mathrm{Fl}_{\mathbf{P}_0}$. One key idea leading to the proof of Theorem 4.5 is the following simple observation.

Lemma 4.12 *The functor e is exact with respect to the perverse t-structure on the source and the usual t-structure on the target.*

Proof Recall from Lemma 3.4 that any object in $\mathcal{D}(\phi)$ is supported on the open subset $V_{\mathbf{P}}\subset\mathrm{Bun}(\mathbf{P}_0,\mathbf{P}^{++}_\infty)$. Fix a base point $\star\in V_{\mathbf{P}}\subset\mathrm{Bun}_G(\mathbf{P}_0,\mathbf{P}^{++}_\infty)$. We identify $\mathrm{Fl}_{\mathbf{P}_0}$ with $\overrightarrow{h}^{-1}_0(\star)\subset\mathrm{Hk}_0$ and thus identify the uniformization map $\omega:\mathrm{Fl}_{\mathbf{P}_0}=L_0G/\mathbf{P}_0\to\mathrm{Bun}_G(\mathbf{P}_0,\mathbf{P}^{++}_\infty)$ with the restriction of the map $\overleftarrow{h}_0$ in diagram 4.1. By construction, $e(C)$ is the stalk at $\star$ of $T(C,A(\phi))=\overrightarrow{h}_{0,!}(\overleftarrow{h}^*_0A(\phi)\otimes C^{\mathrm{Hk}_0})$. By proper base change, we have

$$e(C)\cong\mathrm{H}^*_c(\omega^{-1}(U),\omega^*A(\phi)\otimes C)\cong\mathrm{H}^*(\omega^{-1}(U),\omega^*A(\phi)\otimes C)\tag{4.2}$$

where the second isomorphism is proved by replacing C by its Verdier dual $\mathbb{D}(C)$ in the first isomorphism and using Lemma 3.4(1).

If $C \in D^b_{\mathbf{P}_0}(\mathrm{Fl}_{\mathbf{P}_0})$ is perverse, then $\omega^* A(\phi) \otimes C$ is a perverse sheaf on $\omega^{-1}(U)$. By (4.2) and the t-exactness properties of direct image functors under an affine morphism, it suffices to show that $\omega^{-1}(U) \subset \mathrm{Fl}_{\mathbf{P}_0}$ is affine. However, using the Birkhoff decomposition (3.1) (with the role of 0 and ∞ swapped), we have $\omega^{-1}(U) \cong \Gamma^+_\infty = (\mathrm{Res}_k^{k[t,t^{-1}]} G) \cap \mathbf{P}^+_\infty$, which is an affine algebraic group. □

4.13 Gaitsgory's nearby cycles functor

Let $\mathcal{P} := \mathrm{Perv}_{\mathbf{P}_0}(\mathrm{Fl}_{\mathbf{P}_0}) \subset D^b_{\mathbf{P}_0}(\mathrm{Fl}_{\mathbf{P}_0})$ be the category of perverse sheaves. As in [29, §5.1.2], we consider the parahoric analog of Gaitsgory's nearby cycles functor in [8]

$$Z_{\mathbf{P}_0} : \mathrm{Rep}(\widehat{G}) \to \mathcal{P}.$$

This is an exact central functor (see [1, §2, Definition1]) that admits an action of the inertia group $\mathcal{I}_0$ because it comes from nearby cycles. By [8, Proposition 7], this action factors through the tame quotient $\mathcal{I}_0^t$ and is unipotent. Therefore, the composition functor $e \circ Z_{\mathbf{P}_0} : \mathrm{Rep}(\widehat{G}) \to \mathcal{P} \to \mathrm{Vect}$ lifts to a tensor functor

$$e \circ Z_{\mathbf{P}_0} : \mathrm{Rep}(\widehat{G}) \to \mathrm{Rep}(\mathcal{I}_0^t). \tag{4.3}$$

By [29, Lemma 5.1], there is a canonical $\mathcal{I}_0$-equivariant isomorphism

$$T_0(Z_{\mathbf{P}_0}(V), A(\phi)) \cong (\mathrm{Kl}^V_{\widehat{G},\mathbf{P}}(\phi))|_{\mathrm{Spec}\, F_0^s} \otimes A(\phi)$$

that is compatible with the monoidal structures. Here $\mathcal{I}_0$ acts on $Z_{\mathbf{P}_0}(V)$ by the above remark and on the geometric stalk $(\mathrm{Kl}^V_{\widehat{G},\mathbf{P}}(\phi))|_{\mathrm{Spec}\, F_0^s}$ (F_0^s is a separable closure of the local field F_0). In other words, we have an isomorphism of tensor functors

$$e \circ Z_{\mathbf{P}_0}(-) \cong \mathrm{Kl}_{\widehat{G},\mathbf{P}}(\phi)|_{\mathrm{Spec}\, F_0^s} : \mathrm{Rep}(\widehat{G}) \to \mathrm{Rep}(\mathcal{I}_0^t) \tag{4.4}$$

4.14 Serre quotients of $\mathcal{P}$

The category $\mathcal{P} = \mathrm{Perv}_{\mathbf{P}_0}(\mathrm{Fl}_{\mathbf{P}_0})$ is not closed under the convolution $\overset{\mathbf{P}_0}{*}$ for general $\mathbf{P}_0$. There is a universal way to fix this problem. Let $\mathcal{N} \subset \mathcal{P}$ be the Serre subcategory of $\mathcal{P}$ generated (under extensions) by irreducible objects that appear as simple constituents of ${}^p\mathrm{H}^i(C_1 \overset{\mathbf{P}_0}{*} C_2)$ for some $C_1, C_2 \in \mathcal{P}$ and $i \neq 0$. Consider the Serre quotient $\overline{\mathcal{P}} := \mathcal{P}/\mathcal{N}$.

Lemma 4.15 *The functor ${}^p\mathrm{H}^0(- \overset{\mathbf{P}_0}{*} -) : \mathcal{P} \times \mathcal{P} \to \mathcal{P} \twoheadrightarrow \overline{\mathcal{P}}$ factors through $\overline{\mathcal{P}} \times \overline{\mathcal{P}}$ and defines a monoidal structure $\circledast : \overline{\mathcal{P}} \times \overline{\mathcal{P}} \to \overline{\mathcal{P}}$ on $\overline{\mathcal{P}}$.*

Proof We show that if an irreducible object $C \in \mathcal{N}$, then ${}^p\mathrm{H}^0(C \overset{\mathbf{P}_0}{*} C') \in \mathcal{N}$ for any $C' \in \mathcal{P}$. By definition, C is a subquotient of ${}^p\mathrm{H}^i(C_1 \overset{\mathbf{P}_0}{*} C_2)$ for some $i \neq 0$ and C_1, C_2

simple objects in $\mathcal{P}$. By the decomposition theorem, C is in fact a direct summand of ${}^p\mathrm{H}^i(C_1 \overset{\mathbf{P}_0}{*} C_2)$. Therefore, it suffices to show that ${}^p\mathrm{H}^0({}^p\mathrm{H}^i(C_1 \overset{\mathbf{P}_0}{*} C_2) \overset{\mathbf{P}_0}{*} C') \in \mathcal{N}$. Again by the decomposition theorem, the spectral sequence ${}^p\mathrm{H}^j({}^p\mathrm{H}^i(C_1 \overset{\mathbf{P}_0}{*} C_2) \overset{\mathbf{P}_0}{*} C') \Rightarrow {}^p\mathrm{H}^{i+j}(C_1 \overset{\mathbf{P}_0}{*} C_2 \overset{\mathbf{P}_0}{*} C')$ degenerates at E_2, and therefore, ${}^p\mathrm{H}^0({}^p\mathrm{H}^i(C_1 \overset{\mathbf{P}_0}{*} C_2) \overset{\mathbf{P}_0}{*} C')$ is a subquotient of ${}^p\mathrm{H}^i(C_1 \overset{\mathbf{P}_0}{*} C_2 \overset{\mathbf{P}_0}{*} C')$, hence in $\mathcal{N}$ since $i \neq 0$. This shows that the functor ${}^p\mathrm{H}^0(- \overset{\mathbf{P}_0}{*} -)$ factors through $\overline{\mathcal{P}} \times \overline{\mathcal{P}}$. The associativity of $\circledast$ follows from the associativity of the original convolution product $\overset{\mathbf{P}_0}{*}$. □

4.16 The cell subquotient category

Let a be the length of the longest element of $W_{\mathbf{P}}$, so we have $a = a(\underline{c})$ by Lemma 4.6. Let $\pi : \mathrm{Fl} = L_0G/\mathbf{I}_0 \to \mathrm{Fl}_{\mathbf{P}_0} = L_0G/\mathbf{P}_0$ be the projection. Let $\mathcal{Q} = \mathrm{Perv}_{\mathbf{I}_0}(\mathrm{Fl})$. Since π is smooth of relative dimension a, the functor $\pi^\natural := \pi^*[a](a/2)$ gives an exact functor $\mathcal{P} \to \mathcal{Q}$ that is also fully faithful (we fix a square root of $q = \#k$, hence the half Tate twist makes sense). We may therefore identify $\mathcal{P}$ with a full subcategory of $\mathcal{Q}$ via the functor $\pi^\natural$.

The category $\mathcal{Q}$ carries a filtration by full subcategories indexed by two-sided cells in W_{aff}. Let $\underline{c} = \underline{c}_{\mathbf{P}}$ be the cell containing the longest element in $W_{\mathbf{P}}$. Then $\mathcal{P} \subset \mathcal{Q}_{\leq \underline{c}}$. The category $\mathcal{P}_{<\underline{c}} := \mathcal{P} \cap \mathcal{Q}_{<\underline{c}}$ is a Serre subcategory (generated by those irreducible objects in $\mathcal{P}$ indexed by longest representatives of $(W_{\mathbf{P}}, W_{\mathbf{P}})$-double cosets in W_{aff} that belong to a cell smaller than $\underline{c}$). Let $\mathcal{P}_{\underline{c}} := \mathcal{P}/\mathcal{P}_{<\underline{c}}$ be the Serre quotient. This is a full subcategory of a similar Serre quotient $\mathcal{Q}_{\underline{c}} := \mathcal{Q}_{\leq \underline{c}}/\mathcal{Q}_{<\underline{c}}$. By [20, §2.4–2.5], $\mathcal{Q}_{\underline{c}}$ carries a monoidal structure $\odot$ given by truncated convolution: $\widetilde{C}_1 \odot \widetilde{C}_2$ is the image of ${}^p\mathrm{H}^a(\widetilde{C}_1 \overset{\mathbf{I}_0}{*} \widetilde{C}_2)(a/2)$ in $\mathcal{Q}_{\underline{c}}$.

Lemma 4.17 *The category $\mathcal{P}_{\underline{c}} \subset \mathcal{Q}_{\underline{c}}$ is closed under the truncated convolution $\odot$. The quotient functor $\mathcal{P} \twoheadrightarrow \mathcal{P}_{\underline{c}}$ factors through $\overline{\mathcal{P}}$ and induces a monoidal Serre quotient functor*

$$\eta : (\overline{\mathcal{P}}, \circledast) \twoheadrightarrow (\mathcal{P}_{\underline{c}}, \odot)$$

Proof For $C_1, C_2 \in \mathcal{P}$ with image $\widetilde{C}_1$ and $\widetilde{C}_2$ in $\mathcal{Q}$, we have

$$\pi^\natural C_1 \overset{\mathbf{I}_0}{*} \pi^\natural C_2 \cong \pi^\natural (C_1 \overset{\mathbf{P}_0}{*} C_2) \otimes \mathrm{H}^*(\mathbf{P}_0/\mathbf{I}_0)[a](a/2). \tag{4.5}$$

We first show that $\mathcal{N} \subset \mathcal{P}_{<\underline{c}}$ so the quotient functor factors as $\overline{\mathcal{P}} \twoheadrightarrow \mathcal{P}_{\underline{c}}$. For a simple perverse sheaf $C_w \in \mathcal{N}$ indexed by $w \in W_{\mathbf{P}}\backslash W_{\mathrm{aff}}/W_{\mathbf{P}}$, it appears in ${}^p\mathrm{H}^i(C_1 \overset{\mathbf{P}_0}{*} C_2)$ for some simple perverse sheaves $C_1, C_2 \in \mathcal{P}$ and some $i \neq 0$. Since $C_1 \overset{\mathbf{P}_0}{*} C_2$ is isomorphic to its own Verdier dual, we may assume $i > 0$. Using (4.5) and taking the top cohomology of $\mathrm{H}^*(\mathbf{P}_0/\mathbf{I}_0)$, we conclude that $\pi^\natural\, {}^p\mathrm{H}^i(C_1 \overset{\mathbf{P}_0}{*} C_2)$ is a direct summand of ${}^p\mathrm{H}^{a+i}(\pi^\natural C_1 \overset{\mathbf{I}_0}{*} \pi^\natural C_2)(a/2)$. Therefore, $\pi^\natural C_w$ also appears in ${}^p\mathrm{H}^{a+i}(\pi^\natural C_1 \overset{\mathbf{I}_0}{*}$

$\pi^\natural C_2)(a/2)$. Now $\pi^\natural C_w$ is $\mathrm{IC}_{\widetilde{w}}$ (up to a Tate twist) for $\widetilde{w}$ the longest representative of the coset w. By the definition of the a-function, this means $a(\widetilde{w}) \geq a + i > a$; hence, $\widetilde{w}$ belongs to a smaller cell than $\underline{c}$ (we have a priori $\widetilde{w} \leq \underline{c}$). This shows that $\pi^\natural C_w \in \mathcal{Q}_{<\underline{c}}$, hence $C_w \in \mathcal{P}_{<\underline{c}}$. The conclusion is that $\mathcal{N} \subset \mathcal{P}_{<\underline{c}}$, and we have a Serre quotient functor $\overline{\mathcal{P}} \twoheadrightarrow \mathcal{P}_{\underline{c}}$.

Taking a-th perverse cohomologies of both sides of (4.5), we get

$$^p\mathrm{H}^a(\pi^\natural C_1 \overset{\mathbf{I}_0}{*} \pi^\natural C_2) \cong \bigoplus_{i\geq 0} \pi^{\natural p}\mathrm{H}^i(C_1 \overset{\mathbf{P}_0}{*} C_2) \otimes \mathrm{H}^{2a-i}(\mathbf{P}_0/\mathbf{I}_0)(a/2).$$

For $i > 0$ the corresponding term on the right side lies in $\mathcal{N}$, hence have zero image in $\mathcal{P}_{\underline{c}}$ by the previous discussion. Therefore, $\pi^\natural C_1 \odot \pi^\natural C_2$, defined as the image of $^p\mathrm{H}^a(\pi^\natural C_1 \overset{\mathbf{I}_0}{*} \pi^\natural C_2)(a/2)$ in $\mathcal{Q}_{\underline{c}}$, is equal to the image of $\pi^{\natural p}\mathrm{H}^0(C_1 \overset{\mathbf{P}_0}{*} C_2) \otimes \mathrm{H}^{2a}(\mathbf{P}_0/\mathbf{I}_0)(a) \cong \pi^{\natural p}\mathrm{H}^0(C_1 \overset{\mathbf{P}_0}{*} C_2)$ in $\mathcal{P}_{\underline{c}}$, which is the same as the image of $C_1 \circledast C_2 \in \overline{\mathcal{P}}$ in $\mathcal{P}_{\underline{c}}$. This shows that $\mathcal{P}_{\underline{c}}$ is closed under $\odot$ and that the quotient functor $\overline{\mathcal{P}} \to \mathcal{P}_{\underline{c}}$ is monoidal. □

4.18 Finish of the proof of Theorem 4.5

By Lemma 4.12, e is a t-exact functor, and therefore, $e|_{\mathcal{P}}$ factors through the Serre quotient $\overline{\mathcal{P}}$ by the universal property of $\overline{\mathcal{P}}$. We get a monoidal functor $\overline{e} : (\overline{\mathcal{P}}, \circledast) \to (\mathrm{Vect}, \otimes)$.

Let $\overline{\mathcal{P}}' \subset \overline{\mathcal{P}}$ be the full subcategory consisting of subquotients of the images of the composition $\mathrm{Rep}(\widehat{G}) \xrightarrow{Z_{\mathbf{P}_0}} \mathcal{P} \twoheadrightarrow \overline{\mathcal{P}}$. Denote the functor $\mathrm{Rep}(\widehat{G}) \to \overline{\mathcal{P}}'$ by $\bar{Z}$, which is an exact monoidal functor. Similarly, let $\mathcal{P}'_{\underline{c}} \subset \mathcal{P}_{\underline{c}}$ be the full subcategory consisting of subquotient of images of $\eta \circ \bar{Z}$.

Summarizing, we have the following commutative diagram

$$\begin{tikzcd} \mathrm{Rep}(\widehat{G}) \arrow[r, "\bar{Z}"] & \overline{\mathcal{P}}' \arrow[r, hook] \arrow[d, "\eta'"] & \overline{\mathcal{P}} \arrow[r, "\overline{e}"] \arrow[d, "\eta"] & \mathrm{Vect} \\ & \mathcal{P}'_{\underline{c}} \arrow[r, hook] & \mathcal{P}_{\underline{c}} & \end{tikzcd}$$

Both functors $\bar{Z}$ and $\eta' \circ \bar{Z}$ are exact central functors. Applying a criterion of Bezrukavnikov in [1, Proposition 1] to both functors, we see that both $\overline{\mathcal{P}}'$ and $\mathcal{P}'_{\underline{c}}$ are neutral Tannakian categories over $\overline{\mathbb{Q}}_\ell$ and hence the tensor functor η' between them must be faithful ([3, Proposition 1.4]). However, η' is also a Serre quotient functor. This forces η' to be an equivalence of tensor categories. Let $\widehat{H}$ be the Tannakian group of the neutral Tannakian category $\overline{\mathcal{P}}'$. The exact tensor functor $\bar{Z}$ gives a homomorphism $\widehat{H} \to \widehat{G}$ which is an embedding of algebraic groups because every object in $\overline{\mathcal{P}}'$ is a direct summand of the image of some object in $\mathrm{Rep}(\widehat{G})$. Therefore, we may identify the two rows in the above diagram and obtain a factorization of the fiber functor $e \circ Z_{\mathbf{P}_0}$ as

$$e \circ Z_{\mathbf{P}_0} : \mathrm{Rep}(\widehat{G}) \xrightarrow{\mathrm{Res}^{\widehat{G}}_{\widehat{H}}} \mathrm{Rep}(\widehat{H}) \cong \mathcal{P}'_{\underline{c}} \cong \overline{\mathcal{P}}' \xrightarrow{\overline{e}} \mathrm{Vect}.$$

Note that the category $\mathcal{P}'_{\underline{c}}$ (viewed as a subcategory of $\mathcal{Q}_{\underline{c}}$) is the category $\mathcal{A}_{w_{\mathbf{P}}}$ introduced by Bezrukavnikov in [1, §4.3], where $w_{\mathbf{P}} \in W_{\mathbf{P}}$ is the longest element, and is a Duflo (or distinguished) involution ([17, II, Remark following Proposition 1.4]) by Lemma 4.6. Fix a topological generator $\xi \in \mathcal{I}_0^t$, then ξ acts on the restriction functor $\mathrm{Res}^{\widehat{G}}_{\widehat{H}}$. By Tannakian formalism, the action of ξ on the fiber functor $\overline{e} \circ \mathrm{Res}^{\widehat{G}}_{\widehat{H}} = e \circ Z_{\mathbf{P}_0}$ gives an element $u \in \widehat{G}$ that commutes with $\widehat{H}$ (see [1, Theorem 1]). By (4.4), u is conjugate to the image of ξ under the monodromy representation of the local system $\mathrm{Kl}_{\widehat{G},\mathbf{P}}(\phi)$. Therefore, we need to prove that $u \in \underline{u}_{\mathbf{P}}$; i.e., u corresponds to $\underline{c}_{\mathbf{P}}$ under Lusztig's bijection in Theorem 4.2. But this is exactly what was shown by Bezrukavnikov in [1, Theorem 2]. This completes the proof of Theorem 4.5.

Remark 4.19 In [29], we constructed certain $\widehat{G}$-local systems over $\mathbb{P}^1 - \{0, 1, \infty\}$ over number fields. We formulated several conjectures in [29, §5] concerning the local monodromy and rigidity of the local system constructed there. Our method in proving Theorem 4.5 can be used to prove [29, Conjecture 5.10]. A proof of [29, Conjecture 5.11] will follow from results in the forthcoming work [31]. Finally, by [29, Lemma 5.14], [29, Conjecture 5.13] follows from the other two conjectures mentioned above. Therefore, the local systems constructed in the main theorem of [29] are cohomologically rigid. In particular, the G_2-local system we constructed in [29] is geometrically isomorphic to the one constructed by Dettweiler and Reiter in [4], because both of them have the same monodromy at the three punctures, and Dettweiler–Reiter's local system was shown to be rigid as a GL_7-local system. Details of these arguments will appear elsewhere.

5 Rigidity

In this section, we shall deduce that $\mathrm{Kl}_{\widehat{G},\mathbf{P}}(\mathbf{1}, \phi)$ is cohomologically rigid assuming the knowledge of its monodromy at ∞. We work over the base field $\bar{k}$ without changing notation.

5.1 Cohomological rigidity

Recall that a $\widehat{G}$-local system $E_{\widehat{G}}$ over an open subset X° of a complete smooth connected algebraic curve X is called *cohomologically rigid* if

$$\mathrm{H}^*(X, j_{!*} E^{\mathrm{Ad}}_{\widehat{G}}) = 0$$

where $j : X^{\circ} \hookrightarrow X$ is the inclusion, $E^{\mathrm{Ad}}_{\widehat{G}}$ is the local system on X° associated with the adjoint representation of $\widehat{G}$ on its Lie algebra $\widehat{\mathfrak{g}}$, and $j_{!*}$ simply means the sheaf-theoretic direct image $\mathrm{H}^0 j_*$.

Back to the situation of Sect. 2.10 where $\mathcal{K} = \mathbf{1}$ is trivial. We denote $\mathrm{Kl}_{\widehat{G},\mathbf{P}}(\mathbf{1}, \phi)$ by $\mathrm{Kl}_{\widehat{G},\mathbf{P}}(\phi)$. Let $j : X^\circ \hookrightarrow X$ be the open inclusion, and $i_0, i_\infty : \mathrm{Spec}\,\bar{k} \hookrightarrow X$ be the closed embedding of the two points 0 and ∞.

Proposition 5.2 *Assume $\mathcal{K}$ is trivial and G is split. Assume* $\mathrm{char}(k)$ *is sufficiently large so that $m = m(\mathbf{P})$ is a regular elliptic number for $\mathbb{W}$ (see* Sect. 2.6*). Assume that the properties (1) and (2) in* Sect. 2.9 *about the monodromy of* $\mathrm{Kl}_{\widehat{G},\mathbf{P}}(\phi)$ *at* ∞ *hold. Then we have*

$$\mathrm{H}^*(X, j_{!*}\mathrm{Kl}^{\mathrm{Ad}}_{\widehat{G},\mathbf{P}}(\phi)) = 0.$$

i.e., $\mathrm{Kl}_{\widehat{G},\mathbf{P}}(\phi)$ *is cohomologically rigid.*

Proof Since $\widehat{\mathfrak{g}}^{\mathcal{I}_\infty} = 0$ by property (2) of Sect. 2.9, $\mathrm{Kl}^{\mathrm{Ad}}_{\widehat{G},\mathbf{P}}(\phi)$ admits no global sections, i.e., $\mathrm{H}^0(X^\circ, \mathrm{Kl}^{\mathrm{Ad}}_{\widehat{G},\mathbf{P}}(\phi)) = \mathrm{H}^0(X, j_{!*}\mathrm{Kl}^{\mathrm{Ad}}_{\widehat{G},\mathbf{P}}(\phi)) = 0$. Dually $\mathrm{H}^2_c(X^\circ, \mathrm{Kl}^{\mathrm{Ad}}_{\widehat{G},\mathbf{P}}(\phi)) = \mathrm{H}^2(X, j_{!*}\mathrm{Kl}^{\mathrm{Ad}}_{\widehat{G},\mathbf{P}}(\phi)) = 0$.

The distinguished triangle $j_! \to j_{!*} \to i_{0,*}\mathrm{H}^0 i_0^* \oplus i_{\infty,*}\mathrm{H}^0 i_\infty^* \to j_![1]$ gives an exact sequence

$$0 \to \widehat{\mathfrak{g}}^{\mathcal{I}_0} \oplus \widehat{\mathfrak{g}}^{\mathcal{I}_\infty} \to \mathrm{H}^1_c(X^\circ, \mathrm{Kl}^{\mathrm{Ad}}_{\widehat{G},\mathbf{P}}(\phi)) \to \mathrm{H}^1(X, j_{!*}\mathrm{Kl}^{\mathrm{Ad}}_{\widehat{G},\mathbf{P}}(\phi)) \to 0. \tag{5.1}$$

The Grothendieck–Ogg–Shafarevich formula implies

$$\chi_c(X^\circ, \mathrm{Kl}^{\mathrm{Ad}}_{\widehat{G},\mathbf{P}}(\phi)) = -\mathrm{Swan}_\infty(\widehat{\mathfrak{g}}).$$

Using the vanishing of H^2_c and property (2) of Sect. 2.9, we get

$$\dim \mathrm{H}^1_c(X^\circ, \mathrm{Kl}^{\mathrm{Ad}}_{\widehat{G},\mathbf{P}}(\phi)) = \#\Phi/m.$$

On the other hand, by Theorem 4.5, $\widehat{\mathfrak{g}}^{\mathcal{I}_0}$ is the dimension of the centralizer of a unipotent element u in the unipotent conjugacy class $\underline{u}_{\mathbf{P}}$ in $\widehat{G}$. By Steinberg's theorem, $\dim \widehat{\mathfrak{g}}^{\mathcal{I}_0} = \dim \widehat{\mathfrak{g}}^u = 2\dim \mathcal{B}_u + r$ (where r is the rank of G). By Theorem 4.2(1), $\dim \mathcal{B}_u = a(\underline{c}_{\mathbf{P}}) = \ell(w_{\mathbf{P}})$. Hence, $\dim \mathcal{B}_u = \#\Psi(L_{\mathbf{P}})/2$ ($\Psi(L_{\mathbf{P}})$ is the root system of $L_{\mathbf{P}}$), and

$$\dim \widehat{\mathfrak{g}}^{\mathcal{I}_0} = 2\dim \mathcal{B}_u + r = \#\Psi(L_{\mathbf{P}}) + r = \dim L_{\mathbf{P}}.$$

By the dictionary between stable gradings on Lie algebras and regular elements in Weyl groups given in [9, §4], $L_{\mathbf{P}}$ is isomorphic to the neutral component of the fixed point subgroup of an automorphism of $\mathbb{G}$ given by a lifting of a regular elliptic element $w \in \mathbb{W}$ of order m. By analyzing the action of w on the Lie algebra of $\mathbb{G}$, we see that $\dim L_{\mathbf{P}} = \#\Phi/m$. Therefore, the first two terms in the exact sequence (5.1) both have dimension $\#\Phi/m$, hence the third term vanishes. This proves the vanishing of $\mathrm{H}^*_c(X, j_{!*}\mathrm{Kl}^{\mathrm{Ad}}_{\widehat{G},\mathbf{P}}(\phi))$ in all degrees. □

Remark 5.3 When $\mathbf{P} = \mathbf{I}$, we computed the Euler characteristic of $\mathrm{Kl}^{\mathrm{Ad}}_{\widehat{G},\mathbf{I}}(\phi)$ in [11, Theorem 4] and used this to confirm the predictions about the wild monodromy at ∞ in Sect. 2.9 in the Iwahori case (in this case, these predictions were made in [10]). We expect that similar but more complicated computation could be done for general admissible parahoric subgroups $\mathbf{P}$ to either support or disprove the predictions in Sect. 2.9.

6 Examples: quasi-split unitary groups

When $\widehat{G}$ is a group of classical type, we consider the local system $\mathrm{Kl}^{\mathrm{St}}_{\widehat{G},\mathbf{P}}(\mathcal{K}, \phi)$ over $V^{*,\mathrm{st}}_{\mathbf{P}}$ attached to the $\widehat{G}$-local system $\mathrm{Kl}_{\widehat{G},\mathbf{P}}(\mathcal{K}, \phi)$ and the standard representation of $\widehat{G}$. In this section and the two sections following this one, we give explicit descriptions of $\mathrm{Kl}^{\mathrm{St}}_{\widehat{G},\mathbf{P}}(\mathcal{K}, \phi)$. In each case, we first describe the pair $(L_{\mathbf{P}}, V_{\mathbf{P}})$ in terms of a certain cyclic quiver. Then we express the local system $\mathrm{Kl}^{\mathrm{St}}_{\widehat{G},\mathbf{P}}(\mathcal{K}, \phi)$ as the Fourier transform of an explicit complex on $V_{\mathbf{P}}$.

When G is split of type A, the only admissible standard parahoric subgroup is the Iwahori $\mathbf{I}$. In this case, $\mathrm{Kl}_{\widehat{G},\mathbf{I}}(\mathcal{K}, \phi)$ has been calculated in [11, §3] and is identified with the classical Kloosterman sheaves defined by Deligne. In this section, we consider the case when G is a quasi-split unitary group.

6.1 Linear algebra

Assume $\mathrm{char}(k) \neq 2$. Let (M, q) be a quadratic space over k of dimension n. Let $(\cdot, \cdot) : M \times M \to k$ be the associated symmetric bilinear form $(x, y) = q(x + y) - q(x) - q(y)$. Let K' be the ramified quadratic extension of K with uniformizer $\varpi^{1/2}$. Then there is a Hermitian form h on $M \otimes_k K'$ defined as

$$h(x + \varpi^{1/2} y, z + \varpi^{1/2} w) = (x, y) - \varpi(y, w) + \varpi^{1/2}(y, z) - \varpi^{1/2}(x, w) \tag{6.1}$$

for $x, y, z, w \in M \otimes_k K$.

Let $G = \mathrm{U}(M \otimes K', h)$ be the unitary group of the Hermitian space $(M \otimes K', h)$. This corresponds to the absolute group $\mathbb{G} = \mathrm{GL}(M)$ and a nontrivial outer automorphism σ. The regular elliptic numbers m of $(\mathbb{W}, \sigma)$ are in bijection with divisors $d|n$ or $d|n-1$ (in which case we require $d < n-1$) such that the quotient n/d or $(n-1)/d$ is odd. We have $m = 2n/d$ or $m = 2(n-1)/d$ in the two cases. Since $m/2$ is always odd, we write $m/2 = 2\ell + 1$. For d and m as above, we fix a decomposition

$$M = M_{-\ell} \oplus M_{-1} \oplus M_0 \oplus M_1 \oplus \cdots \oplus M_\ell \tag{6.2}$$

such that $d = \dim M_i \leq \dim M_0 \leq d+1$ (for all $i = \pm 1, \ldots, \pm \ell$), and $(M_i, M_j) = 0$ unless $i + j = 0$. Denote the restriction of q to M_0 by q_0. The pairing $(\cdot, \cdot)$ identifies M_{-j} with M_j^*.

Let $\widetilde{\mathbf{P}}_m \subset G(K)$ be the stabilizer of the lattice chain $\Lambda_\ell \supset \Lambda_{\ell-1} \supset \cdots \supset \Lambda_{-\ell}$ defined as

$$\Lambda_i = \bigoplus_{-\ell \leq j \leq i} M_j \otimes \mathcal{O}_{K'} \oplus \bigoplus_{i < j \leq \ell} M_j \otimes \varpi^{1/2} \mathcal{O}_{K'}.$$

Its reductive quotient is $\widetilde{L}_m = \prod_{i=1}^{\ell} \mathrm{GL}(M_i) \times \mathrm{O}(M_0, q_0)$, where $\mathrm{GL}(M_i)$ acts on M_i by the standard representation and on $M_{-i} = M_i^*$ by the dual of the standard representation. A rank one character sheaf $\mathcal{K}$ on $\widetilde{L}_m$ is a collection of character sheaves $(\mathcal{K}_0, \cdots, \mathcal{K}_\ell)$, where $\mathcal{K}_0$ is a rank one character sheaf on $\mathrm{O}(M_0, q_0)$, and for $i = 1, \cdots, \ell$, $\mathcal{K}_i$ is a Kummer sheaf on $\mathbb{G}_m$, viewed as a rank one character sheaf on $\mathrm{GL}(M_i)$ via the determinant map. In other words, $\mathcal{K}$ is pullback from

$$\widetilde{L}_m^\flat \cong \mathrm{O}(M_0, q_0) \times \prod_{i=1}^{\ell} \mathbb{G}_m \tag{6.3}$$

The subgroup $\mathbf{P}_m \subset \widetilde{\mathbf{P}}_m$, defined as the kernel of $\widetilde{\mathbf{P}}_m \to \widetilde{L}_m \to \mathrm{O}(M_0, q_0) \xrightarrow{\det} \{\pm 1\}$, is an admissible parahoric subgroup of $G(K)$ with $m(\mathbf{P}_m) = m$. The vector space $V_m := V_{\mathbf{P}_m}$ is

$$V_m = \mathrm{Hom}(M_1, M_0) \oplus \cdots \oplus \mathrm{Hom}(M_\ell, M_{\ell-1}) \oplus \mathrm{Sym}^2(M_\ell). \tag{6.4}$$

A better way to think of V_m is the following. We consider the cyclic quiver

$$\begin{tikzcd}
 & M_1 \arrow[dl, "\psi_0"] & M_2 \arrow[l, "\psi_1"'] & \cdots \arrow[l] & M_\ell \arrow[l, "\psi_{\ell-1}"'] \\
M_0 \arrow[dr, "\psi_{-1}"] & & & & \\
 & M_{-1} \arrow[r, "\psi_{-2}"] & M_{-2} \arrow[r] & \cdots \arrow[r, "\psi_{-\ell}"] & M_{-\ell} \arrow[uu, "\psi_\ell"]
\end{tikzcd} \tag{6.5}$$

There is an involution τ acting on such quivers: τ sends $\{\psi_i : M_{i+1} \to M_i\}$ to $\{\psi^*_{-i-1} : M_{i+1} = M^*_{-i-1} \to M^*_{-i} = M_i\}$ (indices understood modulo $2\ell + 1$). Then V_m is the space of τ-invariant cyclic quivers.

When $m = 2$, the cyclic quiver degenerates to a quiver with a single vertex $M = M_0$ and a self-adjoint operator ψ on itself.

6.2 Stable locus

The dual space V_m^* is the space of τ-invariant cyclic quivers of the shape (6.5), except that all arrows are reversed. Let $\phi_i : M_i \to M_{i+1}$ be the arrows. We think of $\phi_\ell : M_\ell \to M_{-\ell}$ as a quadratic form on M_ℓ. An element $\phi = (\phi_{-\ell}, \cdots, \phi_\ell) \in V_m^*$ is stable under the L_m-action if and only if

- All maps ϕ_i have the maximal possible rank;
- We have two quadratic forms on M_0: q_0 and the pullback of ϕ_ℓ to M_0 via the map $\phi_{\ell-1}\cdots\phi_0 : M_0 \to M_\ell$. We denote the second quadratic form by q_0'. Then these quadratic forms are in general position: The pencil of quadrics spanned by q_0 and q_0' is degenerate at n distinct points on $\mathbb{P}^1$.

In particular, when $\dim M_0 = d$ and ϕ is stable, then all ϕ_i are isomorphisms, and ϕ_ℓ has at most a one-dimensional kernel. When $\dim M_0 = d+1$ and ϕ is stable, then ϕ_0 is surjective, $\phi_1, \cdots, \phi_{\ell-1}$ are isomorphisms, and ϕ_ℓ must be nondegenerate.

When $m = 2$, V_2^* is simply the space of quadratic forms on M. An element $\phi \in V_2^*(k)$ is stable under the $L_2 = \mathrm{SO}(M, q)$-action exactly when the quadratic form ϕ is in general position with q (note ϕ itself can be degenerate).

6.3 The scheme $\mathfrak{G}_\lambda$

According to Sect. 3.11, we need to describe the variety $\mathfrak{G}_\lambda = \mathfrak{G}_{\le\lambda}$ for the minuscule $\lambda = (1, 0, \cdots, 0)$ (corresponding to the standard representation of $\widehat{G} = \mathrm{GL}_n$) and the morphism $(f', f'') : \mathfrak{G}_\lambda \to \widetilde{L}_m^\flat \times V_m$.

Let $\mathcal{E} = M \otimes \mathcal{O}_{\widetilde{X}}$ with Hermitian form (into $\mathcal{O}_{\widetilde{X}}$) given by a formula similar to (6.1). There is an increasing filtration F_* on the fiber of $\mathcal{E}$ at ∞ defined by $F_{\le i} = \sum_{j\le i} M_i$. We have the tautological trivializations $\mathrm{Gr}_i^F \mathcal{E} = M_i$. There is a decreasing filtration F^* on the fiber of $\mathcal{E}$ at 0 defined by $F^{\ge i} = \sum_{j\ge i} M_i$. The triple $(\mathcal{E}, F_*, F^*)$ defines the open point of $\mathrm{Bun}_G(\widetilde{\mathbf{P}}_0, \mathbf{P}_\infty^+)$.

The group ind-scheme $\mathfrak{G}$ is the group of unitary automorphisms of $\mathcal{E}|_{\widetilde{X}-\{\pm 1\}}$ that preserve the filtrations F_* and F^* and induce the identity on the associated graded of F_*. The locus $\mathfrak{G}_\lambda$ consists of those $g \in \mathfrak{G} \subset G(F')$ ($F' = k(t^{1/2})$) that have a simple pole at $t^{1/2} = 1$ with residue of rank one. Such g can be written uniquely as

$$g = \frac{t^{1/2}}{t^{1/2}-1}A - \frac{1}{t^{1/2}-1}B$$

for $A, B \in \mathrm{GL}(M)$.

Lemma 6.4 (1) *The scheme $\mathfrak{G}_\lambda$ classifies pairs $(A, B) \in \mathrm{O}(M, q) \times \mathrm{O}(M, q)$ such that*

- $(Ax, By) = (Bx, Ay)$ *for all $x, y \in M$.*
- *A lies in the unipotent radical $U(F_*)$ of the parabolic $P(F_*) \subset \mathrm{O}(M, q)$ preserving the filtration F_* on M; B lies in the parabolic $P(F^*) \subset \mathrm{O}(M, q)$ preserving the filtration F^* on M.*
- *$A - B$ has rank one.*

(2) *The map $(f', f'') : \mathfrak{G}_\lambda \to \widetilde{L}_m^\flat \times V_m$ is given by*

$$f'(A, B) = (B_{0,0}, \det(B_{1,1}), \dots, \det(B_{\ell,\ell}));$$
$$f''(A, B) = (A_{0,1}, A_{1,2}, \dots, A_{\ell-1,\ell}, -B_{\ell,-\ell})$$

in the block presentation of A, B under the decomposition (6.2).

(3) *Consider the morphism* $j : \mathfrak{G}_\lambda \to \mathbb{P}(M)$ *sending* $(A, B) \in \mathfrak{G}_\lambda$ *to the line in* M *that is the image of the rank one endomorphism* $C := I - A^{-1}B \in \mathrm{End}(M)$. *Then* j *is an open embedding.*

Proof (1) The matrix $g = \frac{t^{1/2}}{t^{1/2}-1}A - \frac{1}{t^{1/2}-1}B$ is unitary if and only if $A, B \in \mathrm{O}(M, q)$ and $(Ax, By) = (Bx, Ay)$. The residue of g at $t^{1/2} = 1$ is $A - B$, which hence has rank one.

(2) The value of g at $t^{1/2} = 0$ is B, hence we have the formula for $f'(A, B)$. Expanding g at $t^{1/2} = \infty$ using the local uniformizer $t^{-1/2}$, we get $g = A + t^{-1/2}(A - B) + O(t^{-1})$, which gives the formula for $f''(A, B)$.

(3) The map j is an open embedding since $U(F_*) \times P(F^*)$ is open in $\mathrm{O}(M, q)$. □

For $i = 0, \cdots, \ell$, let $q_{[-i,i]}$ be the restriction of the quadratic form q to $M_{-i} \oplus \cdots \oplus M_i$, and then extended to M by zero on the rest of the direct summands (so that $q_{[-\ell,\ell]} = q$).

Proposition 6.5 (1) *Under the embedding* $j : \mathfrak{G}_\lambda \hookrightarrow \mathbb{P}(M)$, $\mathfrak{G}_\lambda$ *is the complement of the divisors* $q_0 = 0, q_{[-1,1]} = 0, \cdots, q_{[-\ell+1,\ell-1]} = 0$ *and* $q = 0$. *The map* $(f', f'') : \mathbb{P}(M) \supset \mathfrak{G}_\lambda \to \widetilde{L}^\flat_m \times V_m$ *is given by*

$$f'([v]) = \left(R_{v_0}, \frac{q_0(v)}{q_{[-1,1]}(v)}, \frac{q_{[-1,1]}(v)}{q_{[-2,2]}(v)}, \cdots, \frac{q_{[-\ell+2,\ell-2]}(v)}{q_{[-\ell+1,\ell-1]}(v)}, \frac{q_{[-\ell+1,\ell-1]}(v)}{q(v)}\right);$$
$$f''([v]) = \left(\frac{(-, v_{-1})}{q_0(v)}v_0, \frac{(-, v_{-2})}{q_{[-1,1]}(v)}v_1, \ldots, \frac{(-, v_{-\ell})}{q_{[-\ell+1,\ell-1]}(v)}v_{\ell-1}, \frac{(-, v_\ell)}{q(v)}v_\ell\right).$$

where R_{v_0} *denotes the orthogonal reflection on the quadratic space* (M_0, q_0) *in the vector* v_0.

(2) *The local system* $\mathrm{Kl}^{\mathrm{St}}_{\widehat{G},\mathbf{P}_m}(\mathcal{K})$ *on* $V^{*,\mathrm{rs}}_m$ *attached to the unitary group* G, *the admissible parahoric* $\mathbf{P}_m$, *the character sheaf* $\mathcal{K} = \mathcal{K}_0 \boxtimes \mathcal{K}_1 \boxtimes \cdots \boxtimes \mathcal{K}_\ell$ *on* $\widetilde{L}^\flat_m$ *(where* $\mathcal{K}_0$ *is on* $\mathrm{O}(M_0, q_0)$ *and* $\mathcal{K}_i$ *is on* $\mathbb{G}_m$ *for* $i = 1, \cdots, \ell$*) and the standard representation of the dual group* $\widehat{G} = \mathrm{GL}_n$ *is the Fourier transform of the complex* $f''_! f'^* \mathcal{K}[n-1](\frac{n-1}{2})$.

Proof (1) For $(A, B) \in \mathfrak{G}_\lambda$, let $C = I - A^{-1}B$ which has rank one. The condition $(Ax, By) = (Bx, Ay)$ implies $(Cx, y) = (x, Cy)$ (for all $x, y \in M$). This together with the fact that $I - C \in \mathrm{O}(M, q)$ implies that $C(x) = \frac{(x,v)}{q(v)}v$ for some $v \in M$ with $q(v) = (v, v)/2 \neq 0$. In other words, $I - C = R_v$, the orthogonal reflection in the direction of v. Note that $j(A, B) = [v] \in \mathbb{P}(M)$.

We inductively find $A = A_1 \cdots A_{\ell-1}A_\ell$ such that $AR_v \in \overline{P}$. Here A_i is the identity on M_j for all $j \neq i$ and sends $x_i \in M_i$ to $x_i \mod F_{<i}$. The formulae for the action of A_i on M_i are

$$A_\ell(x_\ell) = x_\ell + \frac{(x_\ell, v_{-\ell})}{q_{[-\ell+1,\ell-1]}(v)} v_{\leq \ell-1};$$
$$A_{\ell-1}(x_{\ell-1}) = x_{\ell-1} + \frac{(x_{\ell-1}, v_{-\ell+1})}{q_{[-\ell+2,\ell-2]}(v)} v_{\leq \ell-2};$$
$$\cdots;$$
$$A_1(x_1) = x_1 + \frac{(x_1, v_{-1})}{q_0(v)} v_{\leq 0}.$$

Here we write $v_{\leq i}$ for the projection of v to the factor $\oplus_{j\leq i} M_j$ of M. Taking the blocks $A_{i,i+1}$, we get the formula for $f''([v])$ except for the last entry. The last entry of $f''([v])$ is the negative of the lower-left corner block $B_{\ell,-\ell}$, which is the same as the corner block of R_v. This proves the formula for $f''([v])$.

The action of $B = AR_v$ on the associated graded $M_i = \mathrm{Gr}^i_{F^*} M$ is given by

$$B_{i,i}(x_i) = x_i - \frac{(x_i, v_{-i})}{q_{[-i,i]}(v)} v_i.$$

In particular, $B_{0,0}$ is the reflection R_{v_0} on M_0. Taking determinants, we get the formula for $f'([v])$.

(2) For the coweight λ, we have $\mathrm{IC}_\lambda = \overline{\mathbb{Q}}_\ell[n-1](\frac{n-1}{2})$. Therefore, (2) follows from Proposition 3.12. $\square$

A direct consequence of Proposition 6.5 is the following.

Corollary 6.6 *Let $\phi = (\phi_{-\ell}, \cdots, \phi_\ell) \in V_m^{*,\mathrm{st}}(k)$ be a stable functional. Recall that $\mathfrak{G}_\lambda \cong \mathbb{P}(M) - \cup_{i=0}^{\ell} Q(q_{[-i,i]})$, the complement of the quadrics defined by the $q_{[-i,i]}$. Let $f_\phi : \widetilde{X}^\circ \times \mathfrak{G}_\lambda \to \mathbb{A}^1$ be given by*

$$f_\phi(x, [v]) = \sum_{i=0}^{\ell-1} \frac{(\phi_i v_i, v_{-i-1})}{q_{[-i,i]}(v)} + \frac{\phi_\ell(v_\ell)}{q(v)} x.$$

Note that $\phi_\ell(-)$ is a quadratic form. Let $\pi : \widetilde{X}^\circ \times \mathfrak{G}_\lambda \to \widetilde{X}^\circ$ be the projection. Then we have an isomorphism over $\widetilde{X}^\circ$

$$\mathrm{Kl}^{\mathrm{St}}_{\widehat{G},\mathbf{P}_m}(\mathbf{1}, \phi) \cong \pi_! f_\phi^* \mathrm{AS}_\psi[n-1]\left(\frac{n-1}{2}\right). \tag{6.6}$$

Here we recall that **1** *means the trivial character sheaf.*

Proof We identify $\widetilde{X}^\circ$ with the torus $\mathbb{G}_m^{\mathrm{rot}}$. The action of $\mathbb{G}_m^{\mathrm{rot}}$ on V_m^* is by scaling the last component $\mathrm{Sym}^2(M_\ell^*)$. Let $a_\phi : \mathbb{G}_m^{\mathrm{rot}} \to V_m^{*,\mathrm{st}}$ be given by the $\mathbb{G}_m^{\mathrm{rot}}$-orbit of ϕ. It takes the form $x \mapsto (\phi_0, \cdots, \phi_{\ell-1}, x\phi_\ell)$. By Corollary 3.10 and Proposition 6.5(2),

$$\mathrm{Kl}^{\mathrm{St}}_{\widehat{G},\mathbf{P}_m}(\mathbf{1}, \phi) = a_\phi^* \mathrm{Kl}^{\mathrm{St}}_{\widehat{G},\mathbf{P}_m}(\mathbf{1}) \cong a_\phi^* \mathrm{Four}_\psi(f''_! \overline{\mathbb{Q}}_\ell)[n-1]\left(\frac{n-1}{2}\right).$$

By proper base change, the last term is isomorphic to $\pi_! b_\phi^* \mathrm{AS}_\psi[n-1]$ where $b_\phi : \mathbb{G}_m^{\mathrm{rot}} \times \mathfrak{G}_\lambda \to \mathbb{A}^1$ is given by $(x, [v]) \mapsto x \cdot_{\mathrm{rot}} f''([v])$. The explicit formula of f'' in Proposition 6.5(1) shows that $b_\phi = f_\phi$. Therefore, (6.6) holds. □

When $m = 2$, we may describe the local system $\mathrm{Kl}^{\mathrm{St}}_{\widehat{G},\mathbf{P}}(\mathbf{1}, \phi)$ more explicitly.

Proposition 6.7 *Let $\phi \in V_2^{*,\mathrm{st}}(k)$ such that the pencil of quadrics spanned by q and ϕ is degenerate exactly at $\phi - \lambda_i q$ for distinct elements $\lambda_1, \cdots, \lambda_n \in \bar{k}$.*

(1) *Consider the morphism*

$$f_\phi : \mathbb{P}(M) - Q(q) \to \mathbb{A}^1$$
$$[v] \mapsto \frac{\phi(v)}{q(v)}.$$

The local system $\mathrm{Kl}^{\mathrm{St}}_{\widehat{G},2}(\mathbf{1}, \phi)$ over $\widetilde{X}^\circ \cong \mathbb{G}_m^{\mathrm{rot}}$ is the restriction of the Fourier transform of $f_{\phi,!}\overline{\mathbb{Q}}_\ell[n-1](\frac{n-1}{2})$ from $\mathbb{A}^1$ to $\mathbb{G}_m^{\mathrm{rot}}$.

(2) *When n is odd, $\mathrm{Kl}^{\mathrm{St}}_{\widehat{G},2}(\mathbf{1}, \phi)$ over $\mathbb{G}_m \otimes_k \bar{k}$ is isomorphic to $\oplus_{i=1}^n \mathrm{AS}_{\psi_i}|_{\mathbb{G}_m^{\mathrm{rot}}}$, where $\psi_i(t) = \psi(\lambda_i t)$.*

(3) *When n is even, $\mathrm{Kl}^{\mathrm{St}}_{\widehat{G},2}(\mathbf{1}, \phi)$ over $\mathbb{G}_m \otimes_k \bar{k}$ is isomorphic to $\mathrm{Four}_\psi(\mathcal{L}_\phi[1])|_{\mathbb{G}_m^{\mathrm{rot}}}$, where $\mathcal{L}_\phi$ is the middle extension of the rank one local system on $\mathbb{A}^1 - \{\lambda_1, \cdots, \lambda_n\}$ with monodromy equal to -1 around each puncture λ_i.*

Proof We work over the base field $\bar{k}$ without changing notation. (1) One checks that the map f_ϕ is the composition $\mathbb{P}(M) - Q(q) \xrightarrow{f''} V_2 \xrightarrow{\phi} \mathbb{A}^1$. Therefore, (1) is a direct consequence of Proposition 6.5(2).

(2)(3) The morphism f_ϕ can be compactified into a pencil of quadrics spanned by ϕ and q:

$$\widetilde{f}_\phi : \mathrm{Bl}_Z \mathbb{P}(M) \to \mathbb{P}^1$$

where $Z = Q(\phi) \cap Q(q)$ is the base locus of this pencil and $\mathrm{Bl}_Z\mathbb{P}(M)$ is the blow-up of $\mathbb{P}(M)$ along Z. The fiber of $\widetilde{f}_\phi$ over $[x, y] \in \mathbb{P}^1$ is the quadric defined by $y\phi - xq = 0$. Let $j : \mathbb{P}(M) - Z \hookrightarrow \mathrm{Bl}_Z\mathbb{P}(M)$ be the open immersion and $i : \mathbb{P}^1 \times Z \hookrightarrow \mathrm{Bl}_Z\mathbb{P}(M)$ be the closed immersion of the exceptional divisor. Then $f_{\phi,!}\overline{\mathbb{Q}}_\ell$ is the restriction of $(\widetilde{f}_\phi j)_!\overline{\mathbb{Q}}_\ell$ to $\mathbb{A}^1$. We have an distinguished triangle in $D^b_c(\mathbb{P}^1)$:

$$(\widetilde{f}_\phi j)_!\overline{\mathbb{Q}}_\ell \to \widetilde{f}_{\phi,*}\overline{\mathbb{Q}}_\ell \to \mathrm{H}^*(Z) \otimes \overline{\mathbb{Q}}_{\ell,\mathbb{P}^1} \to (\widetilde{f}_\phi j)_!\overline{\mathbb{Q}}_\ell[1].$$

Therefore, $\widetilde{f}_{\phi,*}\overline{\mathbb{Q}}_\ell|_{\mathbb{A}^1}$ and $f_{\phi,!}\overline{\mathbb{Q}}_\ell$ differ by a constant sheaf, hence their Fourier transforms are the same over $\mathbb{G}_m^{\mathrm{rot}} = \widetilde{X}^\circ$.

Since ϕ is a stable point, the degenerate quadrics $\phi - \lambda_i q$ in the pencil are projective cones over $(n-3)$-dimensional quadrics. All quadrics (degenerate or not) have vanishing odd degree cohomology. Let c_1 be the Chern class of the line bundle $\mathcal{O}_{\mathbb{P}(M)}(1)$, viewed as a morphism $\overline{\mathbb{Q}}_\ell(-1) \to \mathbf{R}^2\widetilde{f}_{\phi,*}\overline{\mathbb{Q}}_\ell$. The i-th power of c_1 gives a morphism

$$c_1^i : \overline{\mathbb{Q}}_\ell(-i) \to \mathbf{R}^{2i}\, \widetilde{f}_{\phi,*}\overline{\mathbb{Q}}_\ell.$$

When n is odd, c_1^i is an isomorphism for $i \neq \frac{n-1}{2}$. When $i = \frac{n-1}{2}$, c_1^i is an injection of sheaves with cokernel a sum of skyscraper sheaves supported at the critical values of $\widetilde{f}_\phi$, i.e., $\{\lambda_1, \cdots, \lambda_n\}$. Therefore, $f_{\phi,!}\overline{\mathbb{Q}}_\ell[n-1]$ differs from the sum of these skyscraper sheaves by constant sheaves. This implies (2).

When n is even, c_1^i is an isomorphism for $i \neq \frac{n-2}{2}$. When $i = \frac{n-2}{2}$, c_1^i is an injection with cokernel $\mathcal{L}_\phi$ being the extension by zero of a rank one local system on $\mathbb{A}^1 - \{\lambda_1, \cdots, \lambda_n\}$. By Picard-Lefschetz theory, $\mathcal{L}_\phi$ has monodromy -1 around $\{\lambda_1, \cdots, \lambda_n\}$, hence $\mathcal{L}_\phi[1]$ is an irreducible perverse sheaf on $\mathbb{A}^1$. Now $f_{\phi,!}\overline{\mathbb{Q}}_\ell[n-1]$ differs from $\mathcal{L}_\phi[1]$ by constant sheaves, which implies (3). □

Finally, we calculate the Euler characteristic of $\mathrm{Kl}^{\mathrm{St}}_{\widehat{G},\mathbf{P}_m}(\mathbf{1}, \phi)$, which then gives evidence for the conjectural description of its Swan conductor at ∞ in Sect. 2.9.

Proposition 6.8 *We have*

$$-\chi_c(\widetilde{X}^\circ, \mathrm{Kl}^{\mathrm{St}}_{\widehat{G},\mathbf{P}_m}(\mathbf{1}, \phi)) = \begin{cases} d & \phi_\ell \text{ nondegenerate;} \\ d-1 & \phi_\ell \text{ degenerate.} \end{cases} \tag{6.7}$$

Proof We shall work over $\bar{k}$ and ignore all Tate twists. □

By the same argument as in [12, Proposition 10.1], the Swan conductor of $\mathrm{Kl}^{\mathrm{St}}_{\widehat{G},\mathbf{P}_m}(\mathcal{K}, \phi)$ at ∞, hence the Euler characteristic of $\mathrm{Kl}^{\mathrm{St}}_{\widehat{G},\mathbf{P}_m}(\mathcal{K}, \phi)$ does not depend on $\mathcal{K}$. Therefore, we assume $\mathcal{K}$ is the trivial character sheaf.

Let $Q_i \subset \mathbb{P}(M)$ be the quadric defined by $q_{[-i,i]} = 0$. Let $U_i = \mathbb{P}(\oplus_{j=-i}^{i} M_j) - \cup_{j=0}^{i} Q_i$. In particular $U_\ell = \mathfrak{G}_\lambda$. We also define $W_i \subset U_i$ to be the quadric defined by $\phi_\ell(\phi_{\ell-1}\cdots\phi_i v_i) = 0$.

Let $f_i : U_\ell \to \mathbb{A}^1$ be the function $[v] \mapsto \frac{(\phi_i v_i, v_{-i-1})}{q_{[-i,i]}([v])}$. This function only depends on the coordinates $v_{-i-1}, \cdots, v_0, \ldots, v_i$. Let $f_{\leq i} = f_0 + f_1 + \cdots + f_i$, and $f_{\leq -1} := 0$.

Consider the projection $\pi_2 : \mathbb{G}_m^{\mathrm{rot}} \times \mathfrak{G}_\lambda \to \mathfrak{G}_\lambda = \mathbb{P}(M) - \cup Q_i$. The stalk of $\pi_{2,!} f_\phi^* \mathrm{AS}_\psi$ over $[v]$ is $f_{\leq \ell-1}^* \mathrm{AS}_\psi \otimes \mathrm{H}^*(\mathbb{G}_m^{\mathrm{rot}}, T_{f_\ell([v])}^* \mathrm{AS}_\psi)$, where $T_{f_\ell([v])}$ is the multiplication by $f_{\ell([v])}$ map $\mathbb{G}_m \to \mathbb{A}^1$. We have $\mathrm{H}^*(\mathbb{G}_m^{\mathrm{rot}}, T_{f_\ell([v])}^* \mathrm{AS}_\psi) = \mathrm{H}_c^*(\mathbb{G}_m)$ if $f_\ell([v]) = 0$ and $\overline{\mathbb{Q}}_\ell[-1]$ if $f_\ell([v]) \neq 0$. Therefore,

$$\begin{aligned}(-1)^{n-1}\chi_c(\widetilde{X}^\circ, \mathrm{Kl}^{\mathrm{St}}_{\widehat{G},\mathbf{P}_m}(\mathbf{1}, \phi)) &= \chi_c(\pi_{2,!} f_\phi^* \mathrm{AS}_\psi) \\ &= -\chi_c(U_\ell, f_{\leq \ell-1}^* \mathrm{AS}_\psi) + \chi_c(W_\ell, f_{\leq \ell-1}^* \mathrm{AS}_\psi).\end{aligned} \tag{6.8}$$

We then show that for $0 \leq i \leq \ell - 1$, we have

$$\chi_c(U_{i+1}, f_{\leq i}^* \mathrm{AS}_\psi) = \chi_c(U_i, f_{\leq i-1}^* \mathrm{AS}_\psi); \tag{6.9}$$
$$\chi_c(W_{i+1}, f_{\leq i}^* \mathrm{AS}_\psi) = \chi_c(W_i, f_{\leq i-1}^* \mathrm{AS}_\psi). \tag{6.10}$$

For (6.9), let $U_i' = \mathbb{P}(M_{-i-1} \oplus M_i) - \cup_{j=0}^i Q_i$. Consider the projection $p : U_{i+1} \to U_i'$ by forgetting the M_{i+1} component. Then we have $\chi_c(U_{i+1}, f_{\le i}^* \mathrm{AS}_\psi) = \chi_c(U_i', f_{\le i}^* \mathrm{AS}_\psi \otimes p_! \overline{\mathbb{Q}}_\ell)$. Fix $[v'] = [v_{-i-1}, \cdots, v_i] \in U_i$, and let $q_i := q_i([v'])$ (which is independent of v_{-i-1}). The fiber of p over $[v']$ is M_{i+1} with the affine hyperplane $(v_{i+1}, v_{-i-1}) + q_i = 0$ removed. When $v_{-i-1} \neq 0$, we have $\mathrm{H}_c^*(p^{-1}([v'])) \cong \mathrm{H}_c^*(\mathbb{G}_m)[-2d+2]$; when $v_{-i-1} = 0$, we have $\mathrm{H}_c^*(p^{-1}([v'])) \cong \overline{\mathbb{Q}}_\ell[-2d]$. Since U_i can be identified with the subscheme of U_i' where $v_{-i-1} = 0$, we have $\chi_c(U_{i+1}, f_{\le i}^* \mathrm{AS}_\psi) = \chi_c(U_i', f_{\le i}^* \mathrm{AS}_\psi \otimes p_! \overline{\mathbb{Q}}_\ell) = \chi_c(U_i, f_{\le i-1}^* \mathrm{AS}_\psi)$ (the function $f_{\le i}$ changes to $f_{\le i-1}$ because $v_{-i-1} = 0$).

For (6.10), let $p : W_{i+1} \to W_i$ is the projection. We have $p_! f_{\le i}^* \mathrm{AS}_\psi = f_{\le i-1}^* \mathrm{AS}_\psi \otimes p_! f_i^* \mathrm{AS}_\psi$, and we need to compare $p_! f_i^* \mathrm{AS}_\psi$ with the constant sheaf on W_i. We decompose p into two steps $W_{i+1} \xrightarrow{p_1} W_i'' \xrightarrow{p_2} U_i$, where $W_i'' \subset \mathbb{P}(M_{-i} \oplus M_{i+1}) - \cup_{j=0}^i Q_i$ is cut out the same the quadric $\phi_\ell(\phi_{\ell-1} \cdots \phi_{i+1} v_{i+1}) = 0$. Fix $v'' = [v_{-i}, \dots, v_{i+1}] \in W_i''$, and let $q_i := q_i([v''])$. The fiber $p_1^{-1}([v'']) = \{v_{-i-1} \in M_{-i-1} | (v_{-i-1}, v_{i+1}) + q_i \neq 0\}$. The function f_i along the fiber $p_1^{-1}([v''])$ is a linear function in v_{-i-1} given by $f_i(v_{-i-1}) = \frac{(\phi_i v_i, v_{-i-1})}{q_i}$. Therefore, the stalk of $p_{1,!} f_i^* \mathrm{AS}_\psi$ at $[v'']$, which is $\mathrm{H}_c^*(p_1^{-1}([v'']), f_i^* \mathrm{AS}_\psi)$, vanishes whenever v_{i+1} is not parallel to $\phi_i v_i$ (or one of them is zero while the other one is not). When $\phi_i v_i \neq 0$, the stalk of $p_{2,!} p_{1,!} f_i^* \mathrm{AS}_\psi$ can be calculated only along those nonzero v_{i+1} parallel to $\phi_i v_i$. Choose an isomorphism $M_{-i-1} \cong \mathbb{A}^d$ such that the first coordinate is given by the functional $(\phi_i v_i, -)/q_i$. Then the stalk of $p_{2,!} p_{1,!} f_i^* \mathrm{AS}_\psi$ is $\mathrm{H}_c^*(\mathbb{G}_m \times \mathbb{A}^1 - \Delta(\mathbb{G}_m), \mathrm{pr}_2^* \mathrm{AS}_\psi)[-2d+2] \cong \overline{\mathbb{Q}}_\ell[-2d+2]$ (here $\Delta(\mathbb{G}_m) \subset \mathbb{G}_m \times \mathbb{A}^1$ is the diagonal). For $\phi_i v_i = 0$, the stalk of $p_{2,!} p_{1,!} f_i^* \mathrm{AS}_\psi$ can be calculated only along $v_{i+1} = 0$, hence equal to $\mathrm{H}_c^*(M_{-i-1}) \cong \overline{\mathbb{Q}}_\ell[-2d]$. Putting together, we conclude that $p_! f_i^* \mathrm{AS}_\psi$ and the constant sheaf $\overline{\mathbb{Q}}_\ell$ are the same in the Grothendieck group of $D_c^b(W_i)$. Therefore, $\chi_c(W_{i+1}, f_{\le i}^* \mathrm{AS}_\psi) = \chi_c(W_i, f_{\le i-1}^* \mathrm{AS}_\psi \otimes p_! f_i^* \mathrm{AS}_\psi) = \chi_c(W_i, f_{\le i-1}^* \mathrm{AS}_\psi)$.

Combining (6.9), (6.10) and (6.8), we see that

$$(-1)^{n-1} \chi_c(\widetilde{X}^\circ, \mathrm{Kl}_{\widehat{G}, \mathbf{P}_m}^{\mathrm{St}}(\mathbf{1}, \phi)) = -\chi_c(U_0) + \chi_c(W_0).$$

Now $U_0 = \mathbb{P}(M_0) - Q_0$ and $W_0 = Q_0' - Q_0$, where Q_0' is the quadric in $\mathbb{P}(M_0)$ defined by $\phi_\ell(\phi_{\ell-1} \cdots \phi_0(v_0)) = 0$. Hence

$$(-1)^{n-1} \chi_c(\widetilde{X}^\circ, \mathrm{Kl}_{\widehat{G}, \mathbf{P}_m}^{\mathrm{St}}(\mathbf{1}, \phi)) = -\chi_c(\mathbb{P}(M_0)) + \chi_c(Q_0) + \chi_c(Q_0') - \chi_c(Q_0 \cap Q_0'). \tag{6.11}$$

Since ϕ is stable, all ϕ_i are surjective for $0 \le i \le \ell - 1$, Q_0' is either minimally degenerate or nondegenerate, and $Q_0 \cap Q_0'$ is always smooth (and equal to the intersection of two smooth quadrics).

In the following table, we list the dimensions of the primitive cohomology of Q_0, Q_0' (always in even degree) and $Q_0 \cap Q_0'$ (in the middle degree, which is $\dim M_0 - 3$) in each case according to whether $\dim M_0 = d$ or $d + 1$, and the parity of d. Using this table and (6.11), one easily deduces (6.7).

$\dim M_0$	Parity of $\dim M_0$	Q_0	Q_0'	$Q_0 \cap Q_0'$
d	Even	1	1 (nondeg) or 0 (degen)	$d-2$
d	Odd	0	0 (nondeg) or 1 (degen)	d
$d+1$	Even	1	0	$d-1$
$d+1$	Odd	0	1	$d+1$

7 Examples: symplectic groups

In this section, we describe explicitly the generalized Kloosterman sheaves for symplectic groups G.

7.1 Linear algebra

Let (M, ω) be a symplectic vector space of dimension $2n$ over k, and let $\mathbb{G} = \mathrm{Sp}(M, \omega)$. Extend ω linearly to a symplectic form on $M \otimes K$, and let $G = \mathrm{Sp}(M \otimes K, \omega)$. Regular elliptic numbers m of $\mathbb{W}$ in this case are in bijection with divisors $d|n$. We have $m = 2n/d$ and set $\ell = n/d$. Fix a decomposition

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_\ell \oplus M_{\ell+1} \oplus \cdots \oplus M_m \tag{7.1}$$

such that $\dim_k M_i = d$ and $\omega(M_i, M_j) = 0$ unless $i + j = m + 1$. Then M_j can be identified with M^*_{m+1-j} using the symplectic form ω.

Let $\mathbf{P}_m \subset G(K)$ be the stabilizer of the chain of lattices $\Lambda_m \supset \Lambda_{m-1} \supset \cdots \supset \Lambda_1$, where

$$\Lambda_i = \sum_{1 \le j \le i} M_j \otimes \mathcal{O}_K + \sum_{i < j \le m} M_j \otimes \varpi \mathcal{O}_K$$

and $\varpi \in \mathcal{O}_K$ is a uniformizer. This is an admissible parahoric subgroup of $G(K)$ with $m(\mathbf{P}_m) = m$. Its Levi quotient is $L_m \cong \prod_{i=1}^{\ell} \mathrm{GL}(M_i)$ where the i-th factor acts on M_i by the standard representation and on $M_{m+1-i} = M_i^*$ by the dual of the standard representation. We have $\widetilde{L}_m = L_m$ and any character sheaf on L_m must factor through the quotient

$$\widetilde{L}_m^\flat = \prod_{i=1}^{\ell} \mathbb{G}_m \tag{7.2}$$

given by the determinants of the GL-factors. We then write the character sheaf $\mathcal{K}$ on L_m as the pullback of $\boxtimes_{i=1}^{\ell} \mathcal{K}_i$ on $\widetilde{L}_m^\flat$, where each $\mathcal{K}_i$ is a Kummer sheaf on $\mathbb{G}_m$. The vector space $V_m := V_{\mathbf{P}_m}$ is

$$V_m = \mathrm{Sym}^2(M_1^*) \oplus \mathrm{Hom}(M_2, M_1) \oplus \cdots \oplus \mathrm{Hom}(M_\ell, M_{\ell-1}) \oplus \mathrm{Sym}^2(M_\ell). \tag{7.3}$$

We may arrange $M_1, \cdots, M_m$ into a cyclic quiver

$$\begin{tikzcd} M_1 \arrow[d, "\psi_m"] & M_2 \arrow[l, "\psi_1"'] & \cdots \arrow[l] & M_\ell \arrow[l, "\psi_{\ell-1}"'] \\ M_m \arrow[r, "\psi_{m-1}"] & M_{m-1} \arrow[r] & \cdots \arrow[r, "\psi_{\ell+1}"] & M_{\ell+1} \arrow[u, "\psi_\ell"'] \end{tikzcd} \tag{7.4}$$

There is an involution τ on the space of such quivers sending $\{\psi_i : M_{i+1} \to M_i\}$ to $\{-\psi^*_{m-i} : M_{i+1} = M^*_{m-i} \to M^*_{m+1-i} = M_i\}$. Then V_m is the set of τ-invariant cyclic quivers of the above shape.

When $m = 2$, the cyclic quiver degenerates to a pair of self-adjoint maps

$$\begin{tikzcd} M_1 \arrow[r, shift left, "\psi_2"] & M_2 \arrow[l, shift left, "\psi_1"] \end{tikzcd}$$

7.2 Stable locus

The dual space V_m^* is the space of τ-invariant cyclic quivers similar to (7.4), except that all the arrows are reversed. Let $\phi_i : M_i \to M_{i+1}$ be the arrows. We think of ϕ_m and ϕ_ℓ as quadratic forms on M_m and M_ℓ, respectively. Then $\phi = (\phi_1, \ldots, \phi_m) \in V_m^*$ is stable if and only if

- All the maps ϕ_i are isomorphisms;
- We have two quadratic forms on M_m: ϕ_m and the transport of ϕ_ℓ to M_m using the isomorphism $\phi_{\ell-1}\cdots\phi_1\phi_m : M_m \xrightarrow{\sim} M_\ell$. They are in general position in the same sense as explained in Sect. 6.2.

7.3 The scheme $\mathfrak{G}_{\le\lambda}$

Let $\lambda \in \mathbb{X}_*(T)$ be the dominant short coroot. The corresponding representation of $\widehat{G} = \mathrm{SO}_{2n+1}$ is the standard representation. We shall describe $\mathfrak{G}_{\le\lambda}$ and the map $(f', f'') : \mathfrak{G}_{\le\lambda} \to \widetilde{L}^\flat_m \times V_m$.

Let $\mathcal{E} = M \otimes \mathcal{O}_X$ be the trivial vector bundle of rank $2n$ over X with a symplectic form (into $\mathcal{O}_X$) given by ω. Define an increasing filtration of the fiber of $\mathcal{E}$ at ∞ by $F_{\le i}\mathcal{E}_\infty = \oplus_{j=1}^{i} M_j$. Define a decreasing filtration of the fiber of $\mathcal{E}$ at 0 by $F^{\ge i}\mathcal{E}_0 = \oplus_{j=i}^{m} M_j$. The group ind-scheme $\mathfrak{G}$ is the group of symplectic automorphisms of $\mathcal{E}|_{X-\{1\}}$ preserving the filtrations F_*, F^* and acting by identity on the associated graded of F_*. The subscheme $\mathfrak{G}_{\le\lambda}$ consists of those $g \in \mathfrak{G} \subset G(F)$ whose entries have at most simple poles at $t = 1$, and $\mathrm{Res}_{t=1} g$ has rank at most one. Therefore, any element in $\mathfrak{G}$ can be written uniquely as

$$g = \frac{t}{t-1}A - \frac{1}{t-1}B$$

for $A, B \in \mathrm{GL}(M)$.

Lemma 7.4 (1) *The scheme* $\mathfrak{G}_{\leq\lambda}$ *classifies pairs* $(A, B) \in \mathrm{Sp}(M, \omega) \times \mathrm{Sp}(M, \omega)$ *satisfying*

- *A lies in the unipotent radical $U(F_*)$ of the parabolic $P(F_*) \subset \mathrm{Sp}(M, \omega)$ preserving the filtration F_* of M; B lies in the opposite parabolic $P(F^*) \subset \mathrm{Sp}(M, \omega)$ preserving the filtration F^* of M.*
- *$A - B$ has rank at most one. Moreover, $C = I - A^{-1}B \in \mathrm{End}(M)$ satisfies $\omega(Cx, y) + \omega(x, Cy) = 0$ for all $x, y \in M$.*

(2) *The map* $(f', f'') : \mathfrak{G}_{\leq\lambda} \to \widetilde{L}^{\flat}_m \times V_m$ *is given by*

$$f'(A, B) = (\det(B_{1,1}), \dots, \det(B_{\ell,\ell}));$$
$$f''(A, B) = (-B_{m,1}, A_{1,2}, \dots, A_{\ell,\ell+1})$$

in the block presentation of A, B under the decomposition (7.1).

(3) *Let* $\mathrm{Sym}^2(M)_{\leq 1} \subset \mathrm{Sym}^2(M)$ *be the subscheme of symmetric pure 2-tensors. Then* $\mathrm{Sym}^2(M)_{\leq 1}$ *can be identified with the scheme of endomorphisms D of M satisfying $\omega(Dx, y)+\omega(x, Dy) = 0$ and of rank at most one). Then the morphism* $j : \mathfrak{G}_{\leq\lambda} \to \mathrm{Sym}^2(M)_{\leq 1}$ *sending* $(A, B) \in \mathfrak{G}_{\leq\lambda}$ *to* $C = I - A^{-1}B$ *is an open embedding.*

Proof (1) The matrix $g = \frac{t}{t-1}A - \frac{1}{t-1}B$ preserves the symplectic form ω if and only if $A, B \in \mathrm{Sp}(M, \omega)$ and $\omega(Ax, By) + \omega(Bx, Ay) = 2\omega(x, y)$ for all $x, y \in M$. The last condition is further equivalent to $\omega(Cx, y)+\omega(x, Cy) = 0$ for all $x, y \in M$. Note that $g(0) = B$, $g(\infty) = A$ and $\mathrm{Res}_{t=1}g = A - B$, hence the rest of the conditions follows.

(2) is proved in the same way as Lemma 6.4(2).

(3) Since D has rank at most one, we may write it as $D(x) = \omega(x, u)v$ for some $u, v \in M$. The condition $\omega(Dx, y)+\omega(x, Dy) = 0$ for all $x, y \in M$ implies that u and v are parallel vectors. Hence, $u\cdot v \in \mathrm{Sym}^2(M)_{\leq 1}$. The fact that j is an open embedding follows from the fact that $U(F_*) \times P(F^*) \hookrightarrow \mathrm{Sp}(M, \omega)$ is an open embedding. □

For $u \cdot v \in \mathrm{Sym}^2(M)_{\leq 1}$, write $u = (u_1, \cdots, u_m)$ and $v = (v_1, \cdots, v_m)$ with $u_i, v_i \in M_i$. Then define

$$\gamma_i(u \cdot v) := \omega(v_{m+1-i}, u_i). \tag{7.5}$$

This is independent of the choice of u, v expressing $u\cdot v$ and therefore defines a regular function on $\mathrm{Sym}^2(M)_{\leq 1}$. Note that $\gamma_i + \gamma_{m+1-i} = 0$ since u and v are parallel.

Proposition 7.5 (1) *Under the open embedding* $j : \mathfrak{G}_{\leq\lambda} \hookrightarrow \mathrm{Sym}^2(M)_{\leq 1}$, $\mathfrak{G}_{\leq\lambda}$ *is the complement of the union of the divisors* $\gamma_1 + \cdots + \gamma_i = 1$ *for* $i = 1, \cdots, \ell$. *The morphism* $(f', f'') : \mathrm{Sym}^2(M)_{\leq 1} \supset \mathfrak{G}_{\leq\lambda} \to \widetilde{L}^{\flat}_m \times V_m$ *is given by*

$$f'(u \cdot v) = \left(\frac{1}{1-\gamma_1}, \frac{1-\gamma_1}{1-\gamma_1-\gamma_2}, \cdots, \frac{1-\gamma_1-\cdots-\gamma_{\ell-1}}{1-\gamma_1-\cdots-\gamma_\ell}\right)$$

$$f''(u\cdot v)=\left(\omega(-,u_m)v_m,\frac{\omega(-,u_{m-1})}{1-\gamma_1}v_1,\frac{\omega(-,u_{m-2})}{1-\gamma_1-\gamma_2}v_2,\dots,\right.$$
$$\left.\frac{\omega(-,u_\ell)}{1-\gamma_1-\dots-\gamma_\ell}v_\ell\right).$$

Here we abbreviate $\gamma_i(u\cdot v)$ by γ_i.

(2) *The local system $\mathrm{Kl}^{\mathrm{St}}_{\widehat{G},\mathbf{P}_m}(\mathcal{K})$ attached to the standard representation of $\widehat{G}=\mathrm{SO}_{2n+1}$, the admissible parahoric subgroup $\mathbf{P}_m$ and the rank one character sheaf $\mathcal{K}=\boxtimes_{i=1}^{\ell}\mathcal{K}_i$ is given by (the restriction to $V_m^{*,\mathrm{st}}$ of) the Fourier transform of the complex $f''_! f'^*\mathcal{K}[2n-1](\frac{2n-1}{2})$.*

Proof (1) Write $C=I-A^{-1}B$ as $\omega(-,u)v$ for parellel vectors $u,v\in M$. We inductively construct $A=A_2\cdots A_m\in U(F_*)$ such that $B=A(I-C)\in P(F^*)$, and that A_i is the identity on M_j, $j\neq i$ and $A_i(x_i)\in x_i+F_{\leq i-1}$ for any $x_i\in M_i$. We may inductively determine

$$\begin{aligned}A_mx_m&=x_m+\frac{\omega(x_m,u_1)}{1-\gamma_1}v_{\leq m-1};\\A_{m-1}x_{m-1}&=x_{m-1}+\frac{\omega(x_{m-1},u_2)}{1-\gamma_1-\gamma_2}v_{\leq m-2};\\&\cdots\\A_2x_2&=x_2+\frac{\omega(x_2,u_{m-1})}{1-\gamma_1-\dots-\gamma_{m-1}}v_1.\end{aligned}$$

Here we write $v_{\leq i}$ for the projection of v to the direct factor $\oplus_{j\leq i}M_j$ of M. Therefore, the entries of A corresponding to $\mathrm{Hom}(M_{i+1},M_i)$ $(1\leq i\leq\ell)$ takes the form

$$A_{i,i+1}=\frac{\omega(-,u_{m-i})}{1-\gamma_1-\dots-\gamma_{m-i}}v_i\in\mathrm{Hom}(M_{i+1},M_i).$$

Using $\gamma_i=-\gamma_{m+1-i}$ to simplify the denominators, we get the desired formula for $f'(u\cdot v)$ except for the first entry. The corner block of $B=A(I-C)$ corresponding to $\mathrm{Hom}(M_1,M_m)$ is the same as the corner block of $I-C$, which takes the form $-\omega(-,u_m)v_m$, whose negative gives the first entry of the formula for $f'(u\cdot v)$.

The matrix $B=A(I-C)$ has block diagonal entries

$$B_{i,i}=\mathrm{id}-\frac{\omega(-,u_{m+1-i})}{1-\gamma_1-\dots-\gamma_{m-i}}v_i\in\mathrm{GL}(M_i).$$

Taking determinants, we get

$$\det(B_{i,i})=\frac{1-\gamma_1-\dots-\gamma_{m+1-i}}{1-\gamma_1-\dots-\gamma_{m-i}}.$$

Using that $\gamma_i=-\gamma_{m+1-i}$, we get the desired formula for $f''(u\cdot v)$.

(2) We only need to apply Proposition 3.12 to the standard representation V_λ of SO_{2n+1}; note that $\mathrm{IC}_\lambda=\overline{\mathbb{Q}}_\ell[2n-1](\frac{2n-1}{2})$ in this case. $\square$

Similar to Corollary 6.6, we have

Corollary 7.6 *Let $\phi = (\phi_1, \dots, \phi_m) \in V_m^{*,\mathrm{st}}(k)$ be a stable functional. Recall that $\mathfrak{G}_{\leq\lambda}$ in this case is $\mathrm{Sym}^2(M)_{\leq 1} - \cup_{i=1}^{\ell}\Gamma_i$ where the divisor Γ_i is defined by the equation $\gamma_1 + \cdots + \gamma_i = 1$ for functions γ_i in (7.5). Let $f_\phi : X^\circ \times \mathfrak{G}_{\leq\lambda} \to \mathbb{A}^1$ be given by*

$$f_\phi(x, u \cdot v) = \omega(\phi_m v_m, u_m)x + \sum_{i=1}^{\ell} \frac{\omega(\phi_i v_i, u_{m-i})}{1 - \gamma_1(u \cdot v) - \cdots - \gamma_i(u \cdot v)}.$$

Let $\pi : X^\circ \times \mathfrak{G}_{\leq\lambda} \to X^\circ$ be the projection. Then we have an isomorphism over X°

$$\mathrm{Kl}^{\mathrm{St}}_{\widehat{G},\mathbf{P}_m}(\mathbf{1}, \phi) \cong \pi_! f_\phi^* \mathrm{AS}_\psi[2n-1]\left(\frac{2n-1}{2}\right).$$

8 Examples: split and quasi-split orthogonal groups

In this section, we describe explicitly the generalized Kloosterman sheaves for split and quasi-split orthogonal groups G.

8.1 Linear algebra

Assume $\mathrm{char}(k) \neq 2$. Let (M, q) be a quadratic space of dimension $2n$ or $2n+1$ over k. Let $(\cdot,\cdot) : M \times M \to k$ be the associated symmetric bilinear form $(x, y) = q(x+y) - q(x) - q(y)$. The regular elliptic numbers of m of the root systems of type B_n, D_n and ${}^2D_n$ are in bijection with

- Type B_n ($\dim M = 2n+1$): divisors $d|n$ (corresponding $m = 2n/d$);
- Type D_n ($\dim M = 2n$): even divisors $d|n$ (corresponding $m = 2n/d$) or odd divisors $d|n-1$ (corresponding $m = 2(n-1)/d$);
- Type ${}^2D_n$ ($\dim M = 2n$): odd divisors $d|n$ (corresponding $m = 2n/d$) or even divisors $d|n-1$ (corresponding $m = 2(n-1)/d$).

We write $m = 2\ell$. Fix a decomposition

$$M = M_0 \oplus M_1 \oplus \cdots M_{\ell-1} \oplus M_\ell \oplus M_{\ell+1} \oplus \cdots \oplus M_{m-1} \tag{8.1}$$

where $\dim M_i = d$ for $i = 1, \cdots, \ell-1, \ell+1, \cdots, m-1$, $\dim M_0$ and $\dim M_\ell$ are either d or $d+1$, and we make sure that when $\dim M = 2n+1$, $\dim M_0$ is even. We see that

- Type B_n: $\dim M_0$ is even and $\dim M_\ell$ is odd;
- Type D_n: $\dim M_0 = \dim M_\ell$ is even;
- Type ${}^2D_n$: $\dim M_0 = \dim M_\ell$ is odd.

The decomposition (8.1) should satisfy $(M_i, M_j) = 0$ unless $i + j \equiv 0 \mod m$. The restriction of q to M_0 and M_ℓ are denoted by q_0 and q_ℓ. The pairing $(\cdot, \cdot)$ induce an isomorphism $M_i^* \cong M_{m-i}$.

Let $M_+ = \oplus_{i>0} M_i$, then $M = M_0 \oplus M_+$ and correspondingly $q = q_0 \oplus q_+$. We use $q_{0,K}$ and $q_{+,K}$ to denote their K-linear extension to $M_0 \otimes K$ and $M_+ \otimes K$, respectively. Define a new quadratic form q'_K on $M \otimes K$ by

$$q'_K = \varpi q_{0,K} \oplus q_{+,K}. \tag{8.2}$$

Let $G = \mathrm{SO}(M \otimes K, q'_K)$. We have

- When $\dim M_0$ is even, choosing a maximal isotropic decomposition $M_0 = M_0^+ \oplus M_0^-$ gives a self-dual lattice $M_0^+ \otimes \mathcal{O}_K \oplus M_0^- \otimes \varpi^{-1}\mathcal{O}_K \oplus M_+ \otimes \mathcal{O}_K \subset M \otimes K$. Therefore, G is a split orthogonal group.
- When $\dim M_0$ is odd, G is a quasi-split orthogonal group of type ${}^2D_n$.

Let $\widetilde{\mathbf{P}}_m \subset G(K)$ be the stabilizer of the lattice chain $\Lambda_{m-1} \supset \Lambda_{m-1} \supset \cdots \supset \Lambda_0$ under $G(K)$, where

$$\Lambda_i = \sum_{0 \le j \le i} M_j \otimes \mathcal{O}_K + \sum_{i < j \le m-1} M_j \otimes \varpi \mathcal{O}_K.$$

Its reductive quotient $\widetilde{L}_m$ is the subgroup of $\mathrm{O}(M_0, q_0) \times \prod_{i=1}^{\ell-1} \mathrm{GL}(M_i) \times \mathrm{O}(M_\ell, q_\ell)$ of index two consisting of $(g_0, \cdots, g_\ell)$ where $\det(g_0) = \det(g_\ell)$. Here the factor $\mathrm{GL}(M_i)$ acts on M_i by the standard representation and on $M_{m-i} = M_i^*$ by the dual of the standard representation. Any rank one character sheaf $\mathcal{K}$ on $\widetilde{L}_m$ is the pullback of a rank one character sheaf from

$$\widetilde{L}_m^\flat \cong \mathrm{O}(M_0, q_0) \times \prod_{i=1}^{\ell-1} \mathbb{G}_m \times \mathrm{O}(M_\ell, q_\ell) \tag{8.3}$$

where the i-th $\mathbb{G}_m$ corresponds to the determinant of $\mathrm{GL}(M_i)$. Note $\widetilde{L}_m \to \widetilde{L}_m^\flat$ is not surjective (cokernel has order two). We may find a character sheaf $\mathcal{K}_0$ on $\mathrm{O}(M_0, q_0)$, a character sheaf $\mathcal{K}_\ell$ on $\mathrm{O}(M_\ell, q_\ell)$, and Kummer sheaves $\mathcal{K}_i$ for $i = 1, \cdots, \ell - 1$, such that $\mathcal{K}$ is the pullback of $\boxtimes_{i=0}^{\ell} \mathcal{K}_i$.

The subgroup $\mathbf{P}_m \subset \widetilde{\mathbf{P}}_m$, defined as the kernel of $\widetilde{\mathbf{P}}_m \to \widetilde{L}_m \to \{\pm 1\}$ by taking the determinant of the first factor, is an admissible parahoric subgroup of $G(K)$ with $m(\mathbf{P}_m) = m$. The vector space $V_m := V_{\mathbf{P}_m}$ is

$$V_m = \mathrm{Hom}(M_1, M_0) \oplus \mathrm{Hom}(M_2, M_1) \oplus \cdots \oplus \mathrm{Hom}(M_\ell, M_{\ell-1}). \tag{8.4}$$

Again, it is more natural to view V_m as τ-invariant cyclic quivers

$$
\begin{tikzcd}
 & M_1 \arrow[dl, "\psi_0"'] & M_2 \arrow[l, "\psi_1"] & \cdots \arrow[l] & M_{\ell-1} \arrow[l, "\psi_{\ell-2}"] & \\
M_0 \arrow[dr, "\psi_{m-1}"'] & & & & & M_\ell \arrow[ul, "\psi_{\ell-1}"'] \\
 & M_{m-1} \arrow[r, "\psi_{m-2}"] & M_{m-2} \arrow[r] & \cdots \arrow[r, "\psi_{\ell+1}"] & M_{\ell+1} \arrow[ur, "\psi_\ell"'] &
\end{tikzcd}
\tag{8.5}
$$

where τ sends $\{\psi_i : M_{i+1} \to M_i\}$ to $\{-\psi^*_{m-1-i} : M_{i+1} = M^*_{m-1-i} \to M^*_{m-i} = M_i\}$ (indices are understood modulo m).

When $m = 2$, the quiver degenerates to a pair of maps

$$
\begin{tikzcd}
M_0 \arrow[r, shift left, "\psi_1"] & M_1 \arrow[l, shift left, "\psi_0"]
\end{tikzcd}
$$

such that $\psi_1 = -\psi_0^*$.

8.2 Stable locus

The dual space V_m^* is the space of τ-invariant cyclic quivers similar to (8.5), except that all the arrows are reversed. Let $\phi_i : M_i \to M_{i+1}$ be the arrows. Then $\phi = (\phi_0, \ldots, \phi_{m-1}) \in V_m^*$ is stable if and only if

- All the maps ϕ_i have the maximal possible rank;
- We have two quadratic forms on M_0: q_0 and the pullback of q_ℓ to M_0 via the map $\phi_{\ell-1} \cdots \phi_0 : M_0 \to M_\ell$. They are in general position in the sense explained in Sect. 6.2.

8.3 The moduli stack

We make a remark about the moduli stack $\mathrm{Bun}_G(\widetilde{\mathbf{P}}_0, \mathbf{P}^+_\infty)$. It classifies the following data

(1) A vector bundle $\mathcal{E}$ over $X = \mathbb{P}^1$ of rank equal to $\dim M$ with a perfect symmetric bilinear pairing on $\mathcal{E}|_{X^\circ}$ (into $\mathcal{O}_{X^\circ}$), such that the corresponding rational map $\iota : \mathcal{E} \dashrightarrow \mathcal{E}^\vee$ has simple pole at 0 whose residue has rank $\dim M_0$ and simple zero at ∞.

(2) A filtration $\mathcal{E}(-\{\infty\}) = F_{-1}\mathcal{E} \subset F_0\mathcal{E} \subset F_1\mathcal{E} \subset \cdots F_{m-1}\mathcal{E} = \mathcal{E}$ together with isomorphisms $\mathrm{Gr}_i^F \mathcal{E} \cong M_i$. Extend this filtration by letting $F_{i+m}\mathcal{E} := F_i\mathcal{E}(\{\infty\})$. Then we require that $F_i\mathcal{E}$ and $F_{2m-1-i}\mathcal{E}$ are in perfect pairing around ∞ under the quadratic form on $\mathcal{E}$, such that the induced pairing between M_i and M_{m-i} is the same as the one fixed in Sect. 8.1.

(3) A filtration $\mathcal{E}(-\{0\}) = F^m\mathcal{E} \subset F^{m-1}\mathcal{E} \subset \cdots \subset F^1\mathcal{E} \subset F^0\mathcal{E} = \mathcal{E}$ such that $\mathrm{Gr}^i_F\mathcal{E}$ has dimension $\dim M_i$. Extend this filtration by letting $F^{i+m}\mathcal{E} :=$

$F^i\mathcal{E}(-\{0\})$. Then we require that $F^i\mathcal{E}$ and $F^{1-i}\mathcal{E}$ are in perfect pairing into around 0 under the quadratic form on $\mathcal{E}$.

(4) A trivialization of $\delta : \mathcal{O}_X \cong \det\mathcal{E}$ such that $\delta^\vee \circ \det(\iota) \circ \delta = t^{-\dim M_0} \in k[t, t^{-1}] = \mathrm{Aut}_{X^\circ}(\mathcal{O}_{X^\circ})$ (which is unique up to a sign).

8.4 The scheme $\mathfrak{G}_\lambda$

Let $\lambda \in \mathbb{X}_*(T)$ be the dominant minuscule coweight such that V_λ is the standard representation of the dual group $\widehat{G} = \mathrm{Sp}_{2n}$ or SO_{2n}. We shall describe the scheme $\mathfrak{G}_\lambda = \mathfrak{G}_{\leq\lambda}$ and the map $(f', f'') : \mathfrak{G}_\lambda \to \widetilde{L}^\flat_m \times V_m$.

Let $\mathcal{E}_+ = M_+ \oplus \mathcal{O}_X$ and $\mathcal{E}_0 = M_0 \otimes \mathcal{O}_X$. Let $q_{0,\mathcal{E}}$ and $q_{+,\mathcal{E}}$ denote the $\mathcal{O}_X$-linear extension of q_0 and q_+ to $\mathcal{E}_0$ and $\mathcal{E}_+$. Define a new rational quadratic form on the trivial vector bundle $\mathcal{E} = M \otimes \mathcal{O}_X = \mathcal{E}_0 \oplus \mathcal{E}_+$ by

$$q_{\mathcal{E}} = t^{-1} q_{0,\mathcal{E}} \oplus q_{+,\mathcal{E}}. \tag{8.6}$$

Then $(\mathcal{E}, q_{\mathcal{E}})$ satisfies the condition in Sect. 8.3(1). The decomposition (8.1) gives the filtrations required in Sect. 8.3(2)(3). Fixing a trivialization of the line $\det M_0 \otimes \det M_\ell$ (which is the same as $\det M$), then we get the data required in Sect. 8.3(4). We thus get a point $(\mathcal{E}, q_{\mathcal{E}}, F_*, F^*, \dots) \in \mathrm{Bun}_G(\widetilde{\mathbf{P}}_0, \mathbf{P}^+_\infty)(k)$. This is the unique point of this moduli stack with trivial automorphism group.

The group ind-scheme $\mathfrak{G}$ is the group of orthogonal automorphisms of $\mathcal{E}|_{X-\{1\}}$ preserving all the auxiliary data specified in Sect. 8.3. The subscheme $\mathfrak{G}_\lambda$ consists of $g \in \mathfrak{G} \subset G(F)$ whose entries have at most simple poles at $t = 1$, and $\mathrm{Res}_{t=1} g$ has rank one. Therefore, any element in $\mathfrak{G}_\lambda$ can be uniquely written as

$$g = \frac{t}{t-1}A - \frac{1}{t-1}B$$

for $A, B \in \mathrm{GL}(M)$.

Lemma 8.5 (1) *The scheme $\mathfrak{G}_\lambda$ classifies pairs $(A, B) \in \mathrm{GL}(M) \times \mathrm{GL}(M)$ of block form*

$$A = \begin{pmatrix} I & A_{0+} \\ 0 & A_{++} \end{pmatrix}; B = \begin{pmatrix} B_{00} & 0 \\ B_{+0} & B_{++} \end{pmatrix} \tag{8.7}$$

under the decomposition $M = M_0 \oplus M_+$ and satisfying

- $A_{++}, B_{++} \in \mathrm{O}(M_+, q_+)$, $B_{00} \in \mathrm{O}(M_0, q_0)$ *and*

$$(x, B_{00}x) = 2q_0(x) + q_+(B_{+0}x); \tag{8.8}$$

$$(A_{++}y, B_{++}y) = q_0(A_{0+}y) + 2q_+(y); \tag{8.9}$$

$$(x, A_{0+}y) = (B_{+0}x, A_{++}y); \tag{8.10}$$

$$(B_{00}x, A_{0+}y) = (B_{+0}x, B_{++}y) \tag{8.11}$$

for all $x \in M_0$ and $y \in M_+$.

- *A_{++} lies in the unipotent radical $U(F_*)$ of the parabolic $P(F_*) \subset \mathrm{SO}(M_+, q_+)$ preserving the filtration F_* of M_+; B_{++} lies in the opposite parabolic $P(F^*) \subset \mathrm{O}(M_+, q_+)$ preserving the filtration F^* of M_+;*
- *$A - B$ has rank one.*

(2) *The morphism $(f', f'') : \mathfrak{G}_\lambda \to \widetilde{L}^\flat_m \times V_m$ is given by*

$$f'(A, B) = (B_{0,0}, \det(B_{1,1}) \cdots, \det(B_{\ell-1,\ell-1}), B_{\ell,\ell});$$
$$f''(A, B) = (A_{0,1}, \dots, A_{\ell-1,\ell})$$

in the block presentation of A, B under the decomposition (8.1).

(3) *Consider the morphism $j : \mathfrak{G}_\lambda \to \mathbb{P}(M)$ sending $(A, B) \in \mathfrak{G}_\lambda$ to the line in M that is the image of the rank one endomorphism $C = I - A^{-1}B$. Then the image of j is contained in the quadric $Q(q)$ defined by $q = 0$, and the resulting map $j : \mathfrak{G}_\lambda \to Q(q)$ is an open embedding.*

Proof (1) The matrix g preserves the quadratic form $q_{\mathcal{E}}$ in (8.6) if and only if the conditions in the first bulleted point hold. The rest of (1) and (2) are similar to their unitary or symplectic counterparts.

(3) Write $C = I - A^{-1}B$ in the block form

$$C = \begin{pmatrix} I - B_{00} + A_{0+}A_{++}^{-1}B_{+0} & A_{0+}A_{++}^{-1}B_{++} \\ -A_{++}^{-1}B_{+0} & I - A_{++}^{-1}B_{++} \end{pmatrix}.$$

Since $I - A_{++}^{-1}B_{++}$ has rank one and $A_{++}^{-1}B_{++} \in \mathrm{O}(M_+, q_+)$, we must have $A_{++}^{-1}B_{++} = R_{v_+}$ (the orthogonal reflection in the direction of v_+) for some nonzero vector $v_+ \in M_+$. In other words, $I - A_{++}^{-1}B_{++} = \frac{(-, v_+)}{q_+(v_+)}v_+$. Since C has rank one, we must have $A_{0+}A_{++}^{-1}B_{++} = \frac{(-, v_+)}{q_+(v_+)}v_0$ for some vector $v_0 \in M_0$. Hence $A_{0+}(x) = -\frac{(x, v_+)}{q_+(v_+)}v_0$. The relation (8.9) implies $q_0(v_0) + q_+(v_+) = 0$; i.e., $v = (v_0, v_+)$ satisfies $q(v) = 0$, or $[v] \in Q(q)$. Note that $[v]$ is exactly the image of C; i.e., $j(A, B) = [v]$.

Using (8.10) and (8.11), we get $B_{00} = R_{v_0}$ and $A_{00}^{-1}B_{+0}(x) = \frac{(x, v_0)}{q_0(v_0)}v_+$. Therefore, $[v]$ determines B_{00}, B_{+0}, A_{0+} and $A_{++}^{-1}B_{++}$. Since $U(F_*) \times P(F^*) \hookrightarrow \mathrm{O}(M_+, q_+)$ is an open embedding, $[v]$ also uniquely determines $A_{++} \in U(F^*)$ and $B_{++} \in P(F^*)$, and j is also an open embedding. □

As in Proposition 6.5, for $1 \le i \le \ell$, we define $q_{[i,m-i]}$ to be the restriction of q to $M_i \oplus \cdots \oplus M_{m-i}$ and extended by zero to M (so that $q_+ = q_{[1,m-1]}$).

Proposition 8.6 *Recall that the group $G = \mathrm{SO}(M \otimes K, q'_K)$ is of type B_n, D_n or ${}^2D_n$.*

(1) *Under the open embedding $j : \mathfrak{G}_\lambda \hookrightarrow Q(q)$, $\mathfrak{G}_\lambda$ is the complement of the union of the divisors defined by $q_\ell = 0, q_{[\ell-1,\ell+1]} = 0, \cdots, q_{[1,m-1]} = 0$. The morphism $(f', f'') : Q(q) \supset \mathfrak{G}_\lambda \to \widetilde{L}_m^\flat \times V_m$ is given by*

$$f'([v]) = \left(R_{v_0}, \frac{q_{[1,m-1]}(v)}{q_{[2,m-2]}(v)}, \frac{q_{[2,m-2]}(v)}{q_{[3,m-3]}(v)}, \dots, \frac{q_{[\ell-1,\ell+1]}(v)}{q_\ell(v)}, R_{v_\ell} \right) \tag{8.12}$$

$$f''([v]) = \left(-\frac{(-, v_{m-1})}{q_{[1,m-1]}(v)} v_0, -\frac{(-, v_{m-2})}{q_{[2,m-2]}(v)} v_1, \dots, -\frac{(-, v_\ell)}{q_\ell(v)} v_{\ell-1} \right). \tag{8.13}$$

Here we write $v = (v_0, \cdots, v_{m-1})$ under the decomposition (8.1).

(2) *The local system $\mathrm{Kl}^{\mathrm{St}}_{\widehat{G},\mathbf{P}_m}(\mathcal{K})$ attached to the standard representation of $\widehat{G} = \mathrm{Sp}_{2n}$ or SO_{2n}, the admissible parahoric subgroup $\mathbf{P}_m$ and the rank one character sheaf $\mathcal{K} = \boxtimes_{i=0}^{\ell} \mathcal{K}_i$ is given by (the restriction to $V_m^{*,\mathrm{st}}$ of) the Fourier transform of the complex $f''_! f'^* \mathcal{K}[\dim M - 2](\frac{\dim M - 2}{2})$.*

Proof (1) In the proof of Lemma 8.5, we have shown that if $j(A, B) = [v]$ and $v = (v_0, v_+)$, then $A_{++}^{-1} B_{++} = R_{v_+}$, $A_{0+}(x) = -\frac{(x,v_+)}{q_+(v_+)} v_0$ and $B_{00} = R_{v_0}$. The last formula gives the first entry of $f'(A, B)$ given in (8.12). The first entry of (8.13) is $A_{0,1}$, which is the first block of $A_{0+} = -\frac{(-,v_+)}{q_+(v_+)} v_0$.

It remains to express R_{v_+} as $A_{++}^{-1} B_{++}$ for $A \in U(F_*)$ and $B \in P(F^*)$. The procedure is the same as in the proof of Proposition 6.5. The expressions of A_{++}^{-1} and B_{++} give the remaining entries in (8.12) and (8.13).

(2) We only need to apply Proposition 3.12 to the standard representation V_λ of $\widehat{G}$; note that $\mathrm{IC}_\lambda = \overline{\mathbb{Q}}_\ell[\dim M - 2](\frac{\dim M - 2}{2})$ in this case. □

Similar to Corollary 6.6, we have

Corollary 8.7 *Let $\phi = (\phi_0, \phi_1, \dots, \phi_{m-1}) \in V_m^{*,\mathrm{st}}(k)$ be a stable functional. Recall that $\mathfrak{G}_\lambda$ in this case is $Q(q) - \cup_{i=1}^{\ell} Q(q_{[i,m-i]})$. Let $f_\phi : \widetilde{X}^\circ \times \mathfrak{G}_\lambda \to \mathbb{A}^1$ be given by*

$$f_\phi(x, [v]) = -\frac{(\phi_0 v_0, v_{m-1})}{q_{[1,m-1]}(v)} x - \sum_{i=1}^{\ell-1} \frac{(\phi_i v_i, v_{m-i-1})}{q_{[i+1,m-i-1]}(v)}.$$

Let $\pi : \widetilde{X}^\circ \times \mathfrak{G}_\lambda \to \widetilde{X}^\circ$ be the projection. Then we have an isomorphism over $\widetilde{X}^\circ$

$$\mathrm{Kl}^{\mathrm{St}}_{\widehat{G},m}(\mathbf{1}, \phi) \cong \pi_! f_\phi^* \mathrm{AS}_\psi[\dim M - 2] \left(\frac{\dim M - 2}{2} \right).$$

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