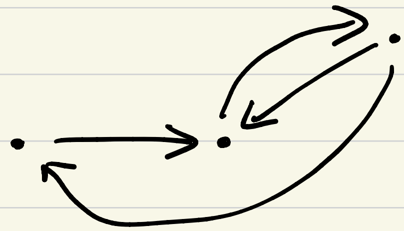
Lecture 8

9/30

- Path algebras of quivers
- group rings

Quiver

Q: 

V = vertices

E = arrows

$s, t : E \to V$.

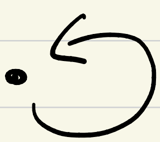
s = source, t = target.

A rep. of Q over k:

$v \in V \rightsquigarrow X_v$ k-v.s.

$e \in E \rightsquigarrow X_{s(e)} \xrightarrow[f_e]{} X_{t(e)}$ k-linear.

Ex:

Q:  Jordan quiver.

Rep of this quiver $= \{ X \circlearrowleft f \}$

$R_Q = k[x]$.

$k = \bar{k}$, simple reps of Q

$\longleftrightarrow X \circlearrowleft f$, no nontrivial f-stable subspaces.

$\longleftrightarrow$ 1-dim'l v.s. $\circlearrowleft f = \lambda \in k$.

Fix n. classify reps of Q with dim $= n$.

$\longleftrightarrow$ $n \times n$ matrices / conjugation.

$\longleftrightarrow$ Jordan canonical forms.

<u>Ex</u>. $\bullet \longrightarrow \bullet$

reps of Q $\longleftrightarrow$ modules for upper triangular matrices.

$$R = \begin{pmatrix} k & k \\ & k \end{pmatrix}$$

Simple rep. $\longleftrightarrow$ simple modules of R

$$\begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} = J(R)$$

$$R/J(R) \simeq k \times k.$$

$$\left.\begin{array}{l} \left(\overset{0}{\bullet} \longrightarrow \overset{k}{\bullet}\right) \\ \left(\overset{k}{\bullet} \longrightarrow \overset{0}{\bullet}\right) \end{array}\right\} \longleftrightarrow \text{2 simple modules}$$

index set = vertices (indep. of k).

<u>Path algebra</u> of Q. R_Q : k-algebra.

s.t. R_Q-modules $\longleftrightarrow$ reps of Q over k,

Construction :

$$R_Q = \bigoplus_{n \geq 0} R_n$$

Assume: V = vertices of Q is <u>finite</u>.

$$R_0 = \underbrace{k \times k \times \cdots \times k}_{V}$$

$$= \bigoplus_{v \in V} k \cdot 1_v \qquad 1_v = (0, \ldots, \overset{\overset{v}{\downarrow}}{1}, \ldots, 0)$$

$$1_v^2 = 1_v.$$

$$R_0 \ni 1 = \sum_{v \in V} 1_v.$$

$$R_1 = \bigoplus_{e \in E} k \cdot x_e \qquad k\text{-v.s. basis } \{x_e\}_{e \in E}$$

$$R_0 \circlearrowright R_1 \circlearrowleft R_0$$

$$\underset{\text{target}}{1_{v'}} \cdot x_e \cdot \underset{\text{source}}{1_v} = \begin{cases} x_e & \text{if } s(e) = v \\ & \text{and } t(e) = v'. \\ 0 & \text{otherwise.} \end{cases}$$

$$R_2 = {}^{R_0 \circlearrowright} R_1 \underset{R_0}{\otimes} R_1 {}^{\circlearrowleft R_0} \qquad R_0 \circlearrowright R_2 \circlearrowleft R_0$$

$$= \bigoplus_{\substack{e_2 \quad e_1 \\ \leftarrow \cdot \leftarrow \\ t(e_1) = s(e_2)}} k \cdot \boxed{x_{e_2} \cdot x_{e_1}}$$

$$R_n = \underbrace{R_1 \underset{R_0}{\otimes} R_1 \underset{R_0}{\otimes} \cdots \underset{R_0}{\otimes} R_1}_{n}$$

$$= \bigoplus_{p=\left(\overset{e_n}{\longleftarrow}\cdot\cdots\cdot\overset{e_1}{\longleftarrow}\right)} k \underbrace{\boxed{x_{e_n} x_{e_{n-1}} \cdots x_{e_1}}}_{x_p}$$

$$R_n \times R_m \xrightarrow{\otimes} R_{n+m}.$$

$$x_p \cdot x_q = x_{pq}$$

R_Q = tensor algebra of R_1 as an R_0-bimodule.

(previously. $A \circlearrowright M.$ ← $T_A(M).$

$R \circlearrowright M \circlearrowleft R$ = ring

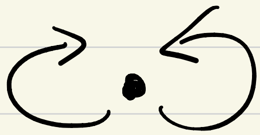
$T_R(M) = R \oplus M \oplus M \otimes_R M \oplus \cdots$).

Exercise. Left R_Q-mods $\overset{\sim}{\longleftrightarrow}$ reps of Q over k

$$x_p \circlearrowright \bigoplus_{v \in V} X_v \longleftarrow\!\shortmid (X_v, f_e)$$

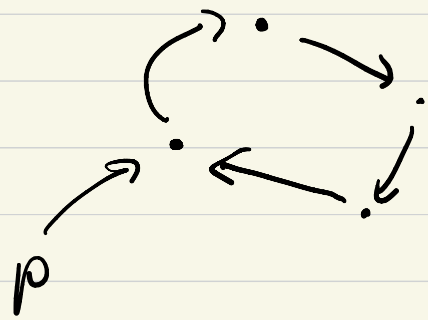
$$x_e = f_e : X_{s(e)} \to X_{t(e)}.$$

$R_0 \ni 1_v$ acts by 1 on X_v, by 0 on $X_{v'}$, $v' \neq v$.

 $\rightsquigarrow R_Q = k\langle \underline{\underline{x}}, \underline{\underline{y}} \rangle$

When is R_Q finite-dim'l / k ?

$\Longleftrightarrow$ * no oriented cycles.



cycle length n

$$\begin{array}{cccc} x_p, & x_p^2, & x_p^3, & \dots \\ \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \\ R_n & R_{2n} & R_{3n}, & \dots \end{array}$$

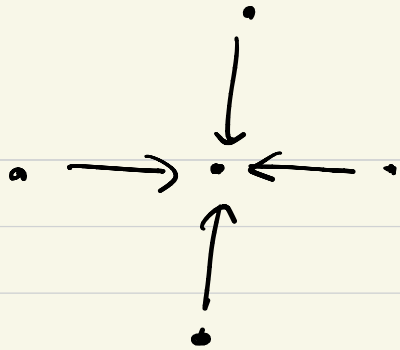
$\Longrightarrow$ if $\exists$ oriented cycle, $\dim_k R_Q = \infty$

if $\nexists$ oriented cycle.

all paths have length $\leq |V|$.

$\Longrightarrow R_n = 0 \quad \forall \; n > |V|$.

$\Longrightarrow \dim_k R_Q < \infty$.



T = tree.

Q = any orientation of T

$\rightsquigarrow$ R_Q is f.d / k.
(artinian).

$J(R_Q) \supseteq R_{\geq 1}$ nilpotent.

$R_Q / R_{\geq 1} = R_0 = \underbrace{k \times \cdots \times k}_{V}$ (semisimple). $\Rightarrow R_{\geq 1} = J(R_Q)$.

simple R_Q-mods ?

$\updownarrow$

simple R_0-mod $\longleftrightarrow \{k_v\}_{v \in V}$.

1-d at v
0 elsewhere.

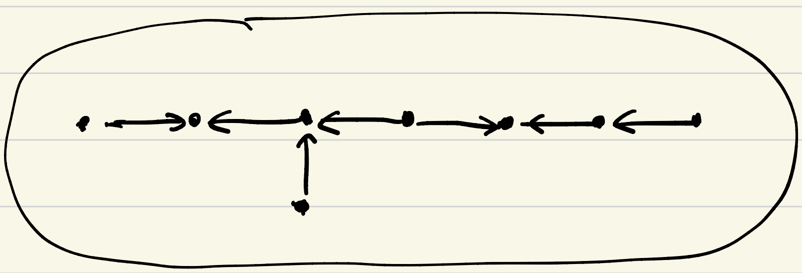
<u>indecomposable reps of Q</u>

not a direct sum.

indecomp $\longleftrightarrow$ $\begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$

$\longleftrightarrow (\lambda, n)$.

$\lambda \in k$, n = size of Jordan block



E_8 Dynkin diagram.

Ex:

$$\begin{tikzcd} & \bullet \arrow[d] & \\ \bullet \arrow[r] & \bullet & \bullet \arrow[l] \end{tikzcd}$$

D_4 finitely many indecomp.

$$\begin{tikzcd} & \bullet \arrow[d] & \\ \bullet \arrow[r] & \bullet & \bullet \arrow[l] \\ & \bullet \arrow[u] & \end{tikzcd}$$

$\widetilde{D}_4$ ∞ indecomp.

$k[G]$ $G =$ finite group.

<u>Thm</u> (Maschke) If $\boxed{\mathrm{char}(k) \nmid |G|}$, then $k[G]$ is semisimple.

$$\Downarrow$$

$$k[G] = \prod_{i \in I} M_{n_i}(D_i)$$

$$\{\text{Irreducible reps of } G \text{ over } k\}$$

$$\|$$

$$\{V_i\}_{i \in I}$$

V_i is a right D_i-v.s. of dim n_i.

$$\dim_k V_i = n_i \cdot \dim_k D_i .$$

D_i f.d. div. alg. / k.

e.g. $k = \mathbb{R}$.

$$\begin{cases} D_i = \mathbb{R}. \quad \mathbb{R}[G] \twoheadrightarrow M_{n_i}(\mathbb{R}). \\ D_i = \mathbb{C}. \\ D_i = \mathbb{H}. \end{cases}$$

(1) $D_i = \mathbb{R}$: $\dim_{\mathbb{R}} V_i = n_i$.

$G \circlearrowright V_i \rightsquigarrow$ G-invariant pos. def. symm. bilinear form

$$(\cdot,\cdot) : V_i \times V_i \longrightarrow \mathbb{R}.$$

$V_{i,\mathbb{C}} = V_i \otimes_{\mathbb{R}} \mathbb{C}$, $(\cdot,\cdot)_{\mathbb{C}}$ symm. bilinear G-invt form on $V_{i,\mathbb{C}}$.

still irreducible (as $\mathbb{C}$-rep. of G)

$$\mathrm{End}_{\mathbb{C}G}(V_{i,\mathbb{C}}) = \underbrace{\mathrm{End}_{\mathbb{R}G}(V_i)} \otimes_{\mathbb{R}} \mathbb{C}$$

$$= \mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}.$$

(2) $D_i = \mathbb{C}$. $\dim_{\mathbb{C}} V_i = n_i$.

$\mathbb{R}[G] \circlearrowright V_i$.

$$V_i \otimes_{\mathbb{R}} \mathbb{C} = V_i \oplus \overline{V_i}$$

$\circlearrowleft G$

$\mathbb{C}$-action is twisted by cx conj.

$V_i \not\cong \overline{V_i}$ as $\mathbb{C}G$-mod.

because: $\mathbb{R}[G] \twoheadrightarrow M_{n_i}(\mathbb{C}) \circlearrowright V_i$

$$\begin{tikzcd}
\mathbb{C}[G] \arrow[r, two heads] \arrow[dddr] & \underbrace{M_{n_i}(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}} \circlearrowright V_i \otimes \mathbb{C}. \\
& M_{n_i}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \arrow[u, equal] \\
& M_{n_i}(\mathbb{C} \times \mathbb{C}). \arrow[u, equal] \\
& M_{n_i}(\mathbb{C}) \times M_{n_i}(\mathbb{C}). \arrow[u, equal]
\end{tikzcd}$$

$$V_i \otimes_{\mathbb{R}} \mathbb{C} = V_i \oplus \overline{V_i}$$

$\Rightarrow$ V_i and $\overline{V_i}$ are ~~dist~~ non-isom simple $\mathbb{C}[G]$-mods.

$\Rightarrow$ no$^{(\neq 0)}$ G-invt $\mathbb{C}$-bilinear form on V_i.

$\overline{V_i} \cong V_i^*$ (always).

~~SIK~~

V_i

③ $D_i = \mathbb{H}$. $\dim_{\mathbb{H}} V_i = n_i$ $V_i \circlearrowleft \mathbb{H}$.

view V_i as $\mathbb{C}$- v.s.

$\Rightarrow$ $\dim_{\mathbb{C}} V_i = 2n_i$.

$(\mathbb{C} \subset \mathbb{H})$.

V_i is an irred $\mathbb{C}$-rep of G of dim $2n_i$.

$h : V_i \times V_i \longrightarrow \mathbb{C}$

(G-invt, >0, hermitian).

$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C} \cdot j \quad \left(\begin{array}{l} j^2 = -1. \\ jz = \bar{z} j \quad \forall z \in \mathbb{C} \end{array} \right)$$

$(u, v) := h(u, vj)$

$\mathbb{C}$- bilinear.

Ex: (u, v) is a G-invt $\mathbb{C}$-bilinear symplectic form on V_i.

$$\begin{cases} D_i = \mathbb{R}. & V_{i,\mathbb{C}}\text{, symm. bilinear } G\text{-invt form} \\ D_i = \mathbb{C}. & V_i \not\cong \overline{V_i}\text{, no form} \\ D_i = \mathbb{H}. & V_i\text{, symp. form} \end{cases}$$

Schur indicator: V: irred $\mathbb{C}$-rep of G.

$$\frac{1}{|G|} \sum_{g \in G} \operatorname{tr}(g^2 | V) = \begin{cases} 1 & \mathbb{R} \\ 0 & \mathbb{C} \\ -1 & \mathbb{H}. \end{cases}$$

$k[G]$. $\quad ch(k) \nmid |G|$.

$$k[G] = \prod_{i \in I} M_{n_i}(D_i) \qquad I \longleftrightarrow \text{irred } k\text{-rep of } G$$

$D_i \rightsquigarrow Z(D_i) =$ field, finite extn of k.

<u>Claim</u>: $L_i = Z(D_i)$ is a separable extn of k.

<u>Pf</u>. $\bar{k}[G]$ is ss.

$$\bar{k}[G] = \prod_{i \in I'} M_{n_i'}(\bar{k}) \qquad I' \longleftrightarrow \text{irred } \bar{k}\text{-rep. of } G.$$

$$M_{n_i}(D_i) \otimes_k \bar{k} = M_{n_i}(D_i \otimes_k \bar{k}).$$

its center $= Z(D_i \otimes_k \bar{k}) = Z(D_i) \otimes_k \bar{k}$

$Z(\bar{k}[G]) = \bar{k} \times \cdots \times \bar{k}$ reduced.

$\Rightarrow Z(M_{n_i}(D_i) \otimes_k \bar{k}) = L_i \otimes_k \bar{k}$ reduced.

$\Longleftrightarrow L_i$ is separable over k. ∎