Lecture 7 9/28

- $J(R)$ and nilpotency.
- Nakayama's Lemma.
- Local ring
- Examples of finite-dim'l k-algebras.

Recall: $I \subset R$ (left / right / 2-sided)

nil-ideal: $\forall\, x \in I$, $x^n = 0$ for some n dep. on x

$\Uparrow$

nilpotent ideal: if $I^n = 0$.

$\iff$ $x_1 x_2 \cdots x_n = 0$ $\forall\, x_1, \dots, x_n \in I$.

<u>Thm</u> R left artinian, then $J(R)$ is a nilpotent ideal.

<u>Cor</u>. R left artinian. Then

- $J(R)$ is the largest left nil-ideal.
- $J(R)$ —— " —— right nil-ideal.
- $J(R)$ —— " —— left nilpotent ideal.
- $J(R)$ —— " —— right
- A left ideal $I \subset R$ is <u>nil</u> $\iff$ <u>nilpotent</u>. (right).

Pf. I nil left ideal (right) $\Rightarrow$ $I \subset J(R)$.
$\Downarrow$
x $\quad 1-x$ is invertible $\quad x \in J(R)$.

$J(R)$ is nil $\Rightarrow$ $J(R)$ is the max nil-ideal (left / right).

and I is nilpotent. ∎

Pf of Thm. $J = J(R)$. $\quad R$ is left artinian $\Rightarrow$

$R \supset J \supset J^2 \supset J^3 \supset \cdots \supset J^n = J^{n+1} = J^{n+2} = \cdots$

want: $J^n = 0$.

Suppose $J^n \neq 0$. $\quad I = J^n$.

$I^2 = I$, $\quad I \neq 0$.

Let $I_0 \subset R$ a min left ideal s.t. $I I_0 \neq 0$.

($I \cdot I = I \neq 0$.)

$I(I I_0) = I^2 \cdot I_0 = I I_0 \neq 0$.

$I I_0 \subset I_0$.

by minimality of I_0 $\Rightarrow$ $I I_0 = I_0$.

Take $a \in I_0$, $a \neq 0$. and $Ia \neq 0$.

minimality of I_0. $\quad Ra = I_0$.

$I I_0 = I_0$

$I \cdot Ra = Ra$.

$\overset{\|}{Ia} \qquad \Downarrow$

$x \cdot a = a \quad$ for some $x \in I$.

$$\underbrace{(1-x)}_{\text{invertible b/c } x \in J(R).} \cdot a = 0.$$

$$\Longrightarrow a = 0.$$

<u>Cor.</u> R left artinian, and R doesn't have $\neq 0$ nilpotent ideal. then R is semisimple.

($\because$ $J(R) = 0$, $R/J(R)$ is ss).

R can have nilpotent elts but no nilp. ideal.

e.g. $M_n(\mathbb{k})$.

<u>Thm</u> (Nakayama) R any ring.

M = f.g. left R-mod.

If $J(R) \cdot M = M$.

$\Longrightarrow M = 0$.

<u>Variant</u>: M f.g. R-mod.

$N \hookrightarrow M$ submod.

$N + J(R) M = M$.

$\left(\Longleftrightarrow \quad N/J(R)N \twoheadrightarrow M/J(R)\cdot M \right)$

Then $N = M$.

In practice, want to show M is gen'd by $x_1, \ldots, x_n$.

(f.g.)

Enough to show $\overline{x_1}, \dots, \overline{x_n} \in M/J(R)\cdot M$
generate it as $R/J(R)$-mod.
(apply Variant with $N = Rx_1 + \dots + Rx_n \subset M$).

Ex. Comm. situations.

A = local ring.
$\hookrightarrow$ A has a unique max ideal. $\mathfrak{m}$.
$A/\mathfrak{m} = k$ (residue field).

- $\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \;\middle|\; (b,p) = 1 \right\}$.

$p\mathbb{Z}_{(p)} \subset \mathbb{Z}_{(p)}$ is the max ideal.

$$p\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \;\middle|\; \begin{matrix} p \mid a \\ (b,p)=1 \end{matrix} \right\}$$

$\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)} \simeq \mathbb{F}_p$.

- $k[[t]] \supset t\,k[[t]]$ = the max ideal.

- $k\langle\langle x_1, \dots, x_n \rangle\rangle$

Nakayama's lemma for comm. local ring A says
M. f.g. A-mod.
$\mathfrak{m}M = M \Rightarrow M = 0$.

e.g. A = noetherian local ring $\supset \mathfrak{m}$.

cotangent space : $\mathfrak{m}/\mathfrak{m}^2$ k-v.s. $\; k = A/\mathfrak{m}$.

noeth $\Rightarrow$ $\mathfrak{m}$ f.g. A-mod.

$\Rightarrow$ $\mathfrak{m}/\mathfrak{m}^2$ f.g. k-v.s.

$\overline{x_1}, \dots, \overline{x_n}$ k-basis of $\mathfrak{m}/\mathfrak{m}^2$.

$x_i \in \mathfrak{m}$ any lifting of $\overline{x_i}$.

Nakayama's lemma $\Rightarrow$ $x_1, x_2, \dots, x_n$ generate $\mathfrak{m}$. (as ideal).

<u>Pf of Thm</u>. M f.g. $\neq 0$.

Want to find a <u>simple</u> quotient of M.

$x_1, \dots, x_n$ generators.

$$\langle x_1 \rangle \subset \langle x_1, x_2 \rangle \subset \cdots \subset \underbrace{\langle x_1, \dots, x_{n-1} \rangle \subset M}_{\text{gen. by } x_n}$$

$$M/\langle x_1, \dots, x_{n-1} \rangle = R/I. \quad (I \text{ left ideal})$$

$I \subset I' =$ max left ideal.

$$M \twoheadrightarrow R/I \twoheadrightarrow R/I' = \text{simple}.$$
$$R/I' = S$$

$M' = \ker(M \twoheadrightarrow S)$

$M' \circlearrowleft J(R)$ $\quad$ $S \circlearrowleft J(R)$

$J(R) \cdot S = 0.$

$\Longleftrightarrow \quad J(R) \cdot M \subset M'.$

Local rings

Def. R is local if it has a unique max left ideal.

TFAE:

(1) R has unique max left ideal;

(2) R —— " —— right —— ;

(3) $R/J(R)$ is a division ring;

(4) $R \setminus R^\times$ is an ideal.

invertible elts

left-right symmetric.

$(1) \Rightarrow (3)$ $\quad I$ = the unique max. left ideal.

$J(R) = I.$

$\bar{R} = R/J(R)$ has no nonzero proper left ideals.

then $\bar{x} \in \bar{R}$, $\bar{x} \neq 0 \Rightarrow \bar{x}$ left invertible

$(\text{o/w } 0 \neq \bar{R} \cdot \bar{x} \subsetneq \bar{R}).$

$\forall \bar{x} \in \bar{R} \setminus 0.$ $\quad \bar{y}\bar{x} = 1.$

$\bar{z}\bar{y} = 1 \quad \Rightarrow \bar{y}$ is invertible with inverse $\bar{x} = \bar{z}$

$\Rightarrow \bar{x}$ is invertible.

$(3) \Rightarrow (1)$. $\quad \bar{R} = R/J(R)$ is a div. ring.

I any $\underline{\underline{\text{max}}}$ left ideal in R.

$J(R) \subset I$.

$\bar{I} \subsetneq \bar{R}$ left ideal.

$\bar{R}$ is div ring $\Rightarrow \bar{I} = 0$.

$\Rightarrow I = J(R)$.

A/m = field.

$R/$(max 2-sided ideal) = simple ring.

$\underline{\text{Ex}}$. $\hat{R} = k\langle\langle x, y\rangle\rangle$.

$$R = k\langle x, y\rangle = \bigoplus_{n \geq 0} R_n$$

$$k\langle\langle x, y\rangle\rangle = \prod_{n \geq 0} R_n$$

$\hat{R}$ is local with $J(\hat{R}) = \prod_{n>0} R_n$.

augm. ideal

$= \ker(k\langle\langle x, y\rangle\rangle \longrightarrow k)$.

Take any $f(x,y) \in \hat{R}$ with nonzero const term

$$f(x,y) = \underline{a_0} + g$$

$\in$ augm. ideal

$$(a_0+g)^{-1} = a_0^{-1}\left(1+a_0^{-1}g\right)^{-1}$$

$$= a_0^{-1}\left(\underline{\underline{1-a_0^{-1}g+a_0^{-2}g^2-\cdots}}\right)$$

$$\qquad\qquad \in R_{\geq 1} \qquad \in R_{\geq 2} \qquad \cdots$$

$$\hat{R}^\times = \hat{R} \setminus (\text{augm. ideal})$$

Finite-dim'l k-algebras R (k is central) (left and right artinian).

Quiver, path algebras,

bimodule

Ex. $R = \begin{pmatrix} \underline{\underline{k}} & \underline{\underline{k}} \\ & \underline{\underline{k}} \end{pmatrix}$ triangular ring.

left R-modules are $\begin{bmatrix} X \\ Y \end{bmatrix}$ X, Y: k-v.s.

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+\boxed{by} \\ dy \end{pmatrix}$$

$a, b, d \in k$ $\quad x \in X$, $y \in Y$.

b really gives a linear map $\underline{\underline{b}}: Y \longrightarrow X$.

$$\Longrightarrow \quad \{\text{left } R\text{-mod}\}$$

$$= \left\{ (X, Y, f) \;\middle|\; \begin{array}{l} X, Y: k\text{-v.s.} \\ f: Y \to X \quad k\text{-linear} \end{array} \right\}$$

$$(X, Y, f) \rightsquigarrow {}^{R \circlearrowright} M = X \oplus Y.$$

$$\begin{pmatrix} a & b \\ & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + b \cdot f(y) \\ dy \end{pmatrix}$$

$$Y \xrightarrow{\ f\ } X$$

$$\boxed{\bullet \longrightarrow \bullet}$$ a quiver = oriented graph.

Q: quiver.

<u>Repr. of Q over k</u> is the assignment:

vertex $v \in V(Q) \rightsquigarrow X_v$: k-v.s.

edge $e = (u \to v)$

$$\rightsquigarrow \quad X_u \xrightarrow{\ f_e\ } X_v$$

(k-linear map)

<u>Claim</u> $\quad Q \rightsquigarrow R_Q$: k-algebra

(path algebra of Q)

s.t.

$$\mathrm{Rep}_k(Q) \cong \text{left } R_Q\text{-mod}.$$