Lecture 6

- Alternative argument for Wedderburn-Artin
- Jacobson radical
- Hopkin-Levitzki Thm (artinian ring $\Rightarrow$ noeth ring).

Thm (Wedderburn-Artin)

R is simple artinian $\iff$ R is semisimple as left R-mod, and is isotypical.
(i.e, $R \cong S^{\oplus n}$)

Pf. $\Leftarrow$

$\Rightarrow$ (Recall. simple R-mod R/I, I = max left ideal).

R = Artinian.

$\Rightarrow$ minimal left ideal exists.
$\neq 0$

$R \circlearrowright I$ = min left ideal.
simple R-mod.

$$IR = \{ \sum_{\text{finite}} x_i y_i \mid x_i \in I,\ y_i \in R \}$$

smallest 2-sided ideal containing I.

R = simple $\Rightarrow$ $IR = R$.

$$\Rightarrow \quad \bigoplus_{y \in R} I \twoheadrightarrow R.$$
$$\qquad\qquad x \longmapsto xy$$

i.e., R is quotient of $\bigoplus I$

$\Rightarrow$ R is semisimple as R-mod, isotypical.

$$R \simeq I^{\oplus n}$$
$$1 \longrightarrow$$

<u>Thm'</u> R is simple artinian

$\Longleftrightarrow$ $R \simeq M_n(D)$.

<u>Pf</u> $\Rightarrow$ M = any simple R-module.

$D = \mathrm{End}_R(M)$. div. ring.

$$R \xrightarrow[\text{ring}]{} \mathrm{End}_D(M).$$

This injective: R is simple $\Rightarrow$ no kernel.

Surjective? R has dense image ✓

If $\dim_D M < \infty$. then dense image

$\Rightarrow$ surjective.

$\Rightarrow$ $R \simeq M_n(D^{op})$

$n = \dim_D M$.

Remains to show: $\dim_D M < \infty$.

Suppose not.

$x_1, x_2, x_3, \ldots$ D-linearly indep vectors in M.

$I_i = \bigcap_{j=1}^{n} \mathrm{Ann}(x_j)$ left ideal in R.

$I_1 \supset I_2 \supset I_3 \supset \cdots$

Why strict? Use density again.

$I_2 \supsetneq I_3$.

$$\begin{array}{cccc} x_1, & x_2, & x_3 & \in M \\ \downarrow r & \downarrow r & \downarrow r & \\ \underset{y_1}{0}, & \underset{y_2}{0}, & y_3 \neq 0. & \end{array}$$

$\exists\, r \in R$. (by density)

then $r \in I_2 \setminus I_3$.

$\Longrightarrow$ contradict with artinian. ∎

Jacobson Radical

$R \overset{??}{\rightsquigarrow}$ semisimple ring.

artinian

$R/J = \overline{R}$ is ss.

Def. $R =$ any ring.

$J(R) =$ intersection of all maximal left ideals.

(compare commutative alg: $J(A) = \bigcap_{\substack{m \subset A \\ \text{max.}}} m$)

Prop $x \in R$. TFAE:

① $x \in J(R)$

② x acts by 0 on any simple left R-mod.

→ ③ $1 - yx$ is left invertible for any $y \in R$.

$\boxed{④ \ 1 - yxz \text{ is invertible for any } y, z \in R}$.

④ is invariant under $R \longleftrightarrow R^{op}$.

$(④ \Longleftrightarrow ①) \Longrightarrow$ $J(R)$ is a two-sided ideal.

$J(R) = J(R^{op})$.

as subset of R.

$$J(R) = \bigcap \begin{pmatrix} \text{max right} \\ \text{ideals} \end{pmatrix}$$

<u>Ex</u>.

① A = f.g. comm. ~~a~~ k-algebra (k = field). $\;\; \big(= k[x_1, \dots, x_n]/I\big)$

$$\underline{J(A)} = \underline{\underline{\mathrm{Nil}(A)}} = \{\text{nilp. elts in } A\}.$$

($\Leftarrow$ Hilbert Nullstellensatz).

$R = M_n(k)$.
{nilp in R} is not closed under $+$
$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

For any R,

Nilpotent $\underset{\text{2-sided}}{\wedge}$ ideal I, $I^n = 0$.

$$I^n = \{\textstyle\sum x_1 x_2 \cdots x_n \mid x_i \in I\}$$

$\Downarrow \quad \cancel{\Uparrow}$

Nil-ideal I: every element in I is nilpotent.

<u>Fact</u>. I is a $\overset{\text{left}}{\vphantom{a}}$ nil-ideal, then $I \subset J(R)$.

Pf. $\forall x \in I$, $\forall y \in R$, $(yx) \in I$. $\quad yx = z$

want: $1 - yx$ is (left) inv.

$yx \to$ nilpotent.

$$(1 + z + z^2 + \cdots)(1 - z) = 1.$$

<u>Ex</u>. $R \subset M_n(k)$
upper triangular.

$J(R) \supseteq$ strictly upper triangular matrices.

$R \longrightarrow$ diagonal matrices $\simeq \underbrace{k \times \cdots \times k}_{n}$

has simple modules $\underline{\underline{S_1, \ldots, S_n}}$.

1st copy of k $\circlearrowright S_1 = k$

$r \in R$ acts by 0 on $S_1, S_2, \ldots, S_n$

$\Longleftrightarrow$ r is strictly upper Δ.

Prop ②. $\Rightarrow$ $J(R) \subset$ strictly upper Δ.

$J(R) =$ strictly upper Δ.

<u>Ex</u>. R. $M_n(R)$.

$$J(M_n(R)) = M_n(J(R)).$$

<u>Proof of prop</u>.

① $\Rightarrow$ ②

$x \in J(R)$ $x \circlearrowright M$ by 0

for any simple M.

$M = R/I$ $I = \underline{\underline{\text{max}}}$ left ideal.

$x : R/I \longrightarrow R/I$

$y \longmapsto x\dot{y} \in R/I$

$xy \underset{?}{\in} I$

② $\Longrightarrow$ ① $\Longrightarrow$ ③ $\Longrightarrow$ ①.

x acts by 0 on simple

$x \in \bigcap I$ max left

$1-yx$ left inv.

② $\Rightarrow$ ①. I max left ideal

$M = R/I$ simple.

$$x: M \xrightarrow{0} M$$

$$1 \longmapsto x = 0 \in R/I$$

$\Rightarrow x \in I$.

① $\Rightarrow$ ③. If ~~not~~ $1-yx$ is not left invertible.

$R\cdot(1-yx)$ is proper left ideal.

$\Rightarrow R\cdot(1-yx) \subset I =$ max left ideal.

$$\left.\begin{array}{l} 1-yx \in I. \\ \text{but } x \in I. \quad yx \in I \end{array}\right\} \Rightarrow 1 \in I$$

③ $\Rightarrow$ ②. $M =$ simple R-mod.

want: $xM = 0$.

If not, $\exists\ m \in M$. s.t. $xm \neq 0$.

M simple, xm generates M.

$\Rightarrow \exists\ y \in R$. s.t. $y(xm) = m$.

$(*)$ $(1-yx)m = 0$.

③: $u(1-yx) = 1$. for some $u \in R$.

$u(*) \Rightarrow m = 0$.

① ② ③ $\Longrightarrow$ ④.

$x \in J(R)$. want: $1 - yxz$ is invertible.

already know: $J(R)$ is 2-sided ideal $\Leftarrow$ ②.

$$J(R) = \bigcap_{M \text{ simple}} \ker(R \to \mathrm{End}(M)).$$

$$= \bigcap (\text{primitive ideals})$$

$yxz \in J(R)$.

③ $\Rightarrow$ $1 - yxz$ is left inv.

$$u(1 - yxz) = 1.$$

$$u = \boxed{1 + uyxz} \quad \text{is left invertible (by ③)}$$

$uyxz \in J(R)$.

$\Rightarrow$ u is invertible

and $1-yxz$ is invertible. ∎

Thm R = left artinian.

$\Rightarrow \overline{R} = R/J(R)$ is semisimple.

Pf. $$J(R) = \bigcap_{\text{max left } I} I$$

artinian $\Longrightarrow$ can find $I_1, \ldots, I_n$ (max left ideals)

s.t. $$J(R) = \bigcap_{i=1}^{n} I_i$$

Idea: $\bar{R}$ ss $\Leftrightarrow {}_{\bar{R}}\bar{R}$ is ss-mod

$\Leftarrow {}_{\bar{R}}\bar{R}$ is a submod of a ss mod.

$$\bar{R} = R/\underbrace{\bigcap_{i=1}^{n} I_i} \hookrightarrow \bigoplus_{i=1}^{n} \underbrace{R/I_i}$$

ss

simple R-mod hence simple $\bar{R}$-mod.

$$\left(R/\bigcap^{\infty} I_i \longrightarrow \prod R/I_i \right).$$

$$J(R) \quad \prod_{i=1}^{r} M_{n_i}(D_i)$$

Thm (Hopkin-Lewitzki)

R is left artinian $\Longrightarrow$ R is left noetherian.

Pf. will show: ${}_R R$ has a composition series.

$(\Longrightarrow {}_R R$ is noetherian.

$$0 \to M' \to M \to M'' \to 0$$

M', M'' noeth $\Leftrightarrow M$ noeth.$)$

$J = J(R)$.

$$\underbrace{R \supset J}\supset \underbrace{J^2 \supset J^3} \supset \cdots \supset J^n = 0.$$

$\Big[$ $\boxed{\text{Next time: } R = \overset{\text{left}}{\text{artinian}} \quad J(R) \text{ is nilpotent}}$ $\Big]$.

$$\overset{ss}{\overset{\|}{\bar{R}}} \circlearrowleft J^i / J^{i+1} = \text{semisimple.}$$

$\bar{R} = R/J$.

also know: J^i/J^{i+1} is artinian left R-mod. (or $\bar{R}$-mod)

$\Downarrow$

J^i/J^{i+1} = finite direct sum of simple modules.

refine $\{J^i\}$ to a composition series.

$\blacksquare$