Proof of Goldie's Theorem

<u>Theorem</u>: R = semiprime, right Noetherian.

(1) S = regular elements in R.
Then S is a right Ore set, and $Q(R) = R_S$ is a semisimple ring.

(2) R is prime $\Longleftrightarrow$ $Q(R)$ is simple Artinian (i.e., $M_n(D)$).
R is a domain $\Longleftrightarrow$ $Q(R)$ is a division ring.

(3). Goldie rank of R = Goldie rank of $Q(R)$.

$Q(R) = \prod M_{n_i}(D_i)$. then

$$\mathrm{grk}(Q(R)) = \sum n_i.$$

Last time:

- $M \subseteq_e N$.

  essential right ideals $I \subset_e R$.

- Goldie rank of a right R-mod M.

<u>Lemma</u>. $M_1 \oplus M_2 \oplus \cdots \oplus M_n \subset_e M$.
each M_i is uniform.

$\Rightarrow \quad grk(M) = n.$

Warning: may happen $M \twoheadrightarrow M''$. $grk(M) < grk(M'')$

e.g. $R = \mathbb{Z}$, $M = \mathbb{Z}$. $\mathbb{Z}/6 = \mathbb{Z}/2 \oplus \mathbb{Z}/3$

Cor 1) $M' \subset M$.

$grk(M') \leq grk(M)$.

2). Suppose M has finite Goldie rk.

$M' \subset M$.

Then $grk(M') = grk(M)$

$\Longleftrightarrow M' \subset_e M$.

(If M' is not ess in M.

$$\begin{array}{ccc} M' & \oplus & N \subset M. \\ \cup & & \neq 0 \\ M_1' \oplus \cdots \oplus M_n' & & \end{array}$$

$\Rightarrow \quad grk(M') < grk(M)$.

If $M' \subset_e M$.

$grk(M') = n \Rightarrow$

$$\underbrace{\boxed{M_1' \oplus \cdots \oplus M_n'}}_{\text{uniform}} \subset_e M' \subset_e \boxed{M}$$

ess

Lemma $\Rightarrow \quad grk(M) = n$.

Fact $M \rightsquigarrow$ injective hull $I(M)$.

when is $I(M)$ indecomposable ?

M is uniform $\iff$ $I(M)$ is indecomp.

$\operatorname{grk}(M) = n$ $\Rightarrow$ $I(M)$ = direct sum of n indecomp. injectives.

Pf of Goldie's Thm assuming:

<u>Lemma</u> R = semiprime, right Noetherian
$I \subset_e R$ ess right ideal.
Then I contains a regular element of R.
(to be proved later).

(1) Check right Ore condition for S = regular elements.
$\boxed{sR \cap aS \neq \emptyset}$ $\forall$ $s \in S$, $a \in R$.

sR right ideal in R.
as R-mod. $R \xrightarrow[\sim]{s} sR$

$\Rightarrow$ $\operatorname{grk}(sR) = \operatorname{grk}(R) < \infty$.

Cor $\Rightarrow$ $sR \subset_e R$.

Want to find $a(s') \in sR$.

$$\begin{array}{ccc} L_a^{-1}(sR) & \subset_e & R \\ \downarrow & \Uparrow & \downarrow \\ sR & \subset_e & R \end{array}$$

right ideal

L_a = left mult by a.
(right R-linear).

want to find a regular elt here.

$L_a^{-1}(sR) \subset_e R$ $\xRightarrow{\text{Lemma}}$ $L_a^{-1}(sR)$ contains a regular element s'.

$as' \in sR$.

Remains to show: $Q(R) = R_S$ is semisimple.

Recall: A ring R' is ss

$\iff$ R' doesn't have proper <u>essential</u> right ideal

Will check: $Q(R) \underset{e}{\supset} J$ essential right ideal

then $J = Q(R)$.

Pf: $I = J \cap R \subset R$.

<u>Claim</u>: $I \underset{e}{\subseteq} R$.

(If not, $I \oplus I' \subset R$.

$\Rightarrow \underbrace{I_S}_{\substack{\| \\ J}} \oplus \underbrace{I'_S}_{\substack{\not\parallel \\ 0}} \subset R_S = Q(R)$.)

Lemma $\Rightarrow$ I contain some $s \in S$.

$\Rightarrow J \ni s$ is invertible in $Q(R)$.

$\Rightarrow J = Q(R)$.

<u>Localization of modules</u>: S = right Ore set $\subset R$.

$$\left.\begin{aligned} M_S &= \varinjlim_{s \in \Sigma} (M \cdot s^{-1}) \circlearrowleft R_S \\ M_S &= M \otimes_R R_S. \end{aligned}\right\} \quad (-)_S \text{ is exact.}$$

$S \subseteq$ regular elements.

$$\begin{array}{ccc} R & \subset & R_S. \\ \cup & & \cup \\ I & \subset & I_S \end{array}$$

$J \subset R_S$. right ideal.

$I = R \cap J$, $\quad I_S \subseteq J$

$\supseteq$ any elt in J as^{-1} $\quad a \in R,\ s \in S$

$\Rightarrow a \in J \cap R = I$

$\Rightarrow as^{-1} \in I_S$.

(2) R is domain $\Longleftrightarrow Q(R)$ is div. ring.

$\Rightarrow$ $S_0 = R \setminus \{0\}$.

$0 \neq as^{-1} \in Q(R)$.

$s, a \in S$. $\Rightarrow$ a and s^{-1} are both invertible in $Q(R)$

$\Rightarrow as^{-1}$ is invertible.

$\Leftarrow$ $R \subset Q(R) = \text{domain}$.

R is prime $\Longleftrightarrow Q(R) \simeq M_n(D)$.

$\Rightarrow$ Suppose $Q(R) = Q_1 \times Q_2$.

$I_i = (Q_i \times \{0\}) \cap R$. $\quad i = 1, 2$.

$I_1 I_2 = 0$.

Observation: $R \subseteq_e Q(R) \circlearrowleft R$

(essential right R-submod)

$\forall\ 0 \neq M \subset Q(R)$
$\circlearrowleft_R$.

$a \in R,\ 0 \neq as^{-1} \in M$. $\Rightarrow 0 \neq a \in M$.

$\Rightarrow 0 \neq a \in M \cap R$.

$\Downarrow$

$I_1 \neq 0,\ I_2 \neq 0$. $\Rightarrow$ R is not prime.

$\Leftarrow$ $Q(R) \cong M_n(D)$. want to show R is prime.

Suppose $aRb = 0$ $a, b \in R$.

$a \neq 0$.

$a \in Q(R) = M_n(D)$ simple ring.

$Q(R)\, a\, Q(R) = Q(R)$.

$\Rightarrow \exists\ x_i, y_i \in Q(R)$. s.t.

$$\sum_{i=1}^{N} x_i a y_i = 1.$$

find common denominator s for $\{y_i\}$

$y_i = z_i \cdot s^{-1}$. some $s \in S$.

$z_i \in R$.

$$\sum x_i a z_i = s.$$

$$0 = \sum x_i \boxed{a z_i b} = \underset{\text{reg}}{s}\, b \Rightarrow b = 0.$$

(3). $\operatorname{grk}_R(R) = \operatorname{grk}_{Q(R)}(Q(R))$. (right ideal)

$R \subset_e Q(R)$ (as R-submod) $\quad \operatorname{grk}_R(R) = \operatorname{grk}_R(Q(R)) = \operatorname{grk}_{Q(R)}(Q(R))$

$$\operatorname{grk}_R(R) = \operatorname{grk}_R(Q(R)) \geq \operatorname{grk}_{Q(R)}(Q(R))$$

want. $\underset{= n}{\operatorname{grk}_R(R)} \leq \operatorname{grk}_{Q(R)}(Q(R))$.

$$I_1 \oplus \cdots \oplus I_n \subset R$$

$$(I_1)_S \oplus \cdots \oplus (I_n)_S \subset R_S = Q(R)$$

Lemma: R = semiprime, right Noe.
$I \subset_e R$, (right $\overset{\text{ess}}{\text{ideal}}$) $\Rightarrow$ I contains a regular element.

Singular submod

$M \circlearrowleft R$.

$\mathrm{Sing}(M) = \{ x \in M \mid \mathrm{Ann}(x) \subseteq_e R \}$.

right ideal

submod

$$\begin{array}{ccc} \mathrm{Ann}(\cancel{x}a) & \subset_e & R \\ \downarrow & & \downarrow L_a \\ \mathrm{Ann}(x) & \subset_e & R \end{array}$$

$M \xrightarrow{f} N$, $\quad \mathrm{Sing}(M) \xrightarrow{f} \mathrm{Sing}(N)$.

$\mathrm{Sing}(R_R) \subset R$

ideal.

right ideal ✓

left ideal ✓.

$\boxed{R = \text{nonsingular if } \mathrm{Sing}(R_R) = 0.}$

Fact: A = comm, $\quad A$ is nonsingular $\Leftrightarrow$ A is reduced.

Fact. R is reduced, then R is nonsingular.

Lemma. R right Noeth. then $\mathrm{Sing}(R_R)$ is nilpotent.
($I = \mathrm{Sing}(R_R)$. $R\mathrm{Ann}(I^m)$ stabilizes ....).

Cor. R semiprime, right Noeth $\Rightarrow$ R is nonsingular.

i.e. $\forall\ 0 \neq a \in R$. $RAnn(a)$ is not ess. in R.

$\iff$ Any essential right $I \subset_e R$

then $LAnn(I) = 0$.

Cor. If $a \in R$ is right regular, $\left(\begin{array}{c} ax=0 \\ \Rightarrow x=0 \end{array} \right)$.

then a is regular.

Pf. $aR \cong R$.

$grk(aR) = grk(R) < \infty$.

$aR \subset_e R$

$xa = 0 \Rightarrow x \in LAnn(\underbrace{aR}_{ess}) = 0$. ∎

Sublemma $I \subset R$ right ideal.

There exists $x \in I$ s.t.

$RAnn(x) \cap I = 0$.

Pf of Lemma $I \subset_e R$.

Sublemma $\Rightarrow \exists\ x \in I$ st.

$RAnn(x) \cap I = 0$.

$\Rightarrow RAnn(x) = 0$.

i.e. x is $\overset{\text{right}}{\text{regular}} \Rightarrow x$ is reg.

Pf of sublemma   Induction on $\mathrm{grk}_R(I)$.

$\mathrm{grk}_R(I) = 1$. $I = \text{uniform} \neq 0$.

R semiprime $\Rightarrow$ $I^2 \neq 0$.

$x, y \in I$

$xy \neq 0$.

Claim $\underbrace{\mathrm{RAnn}(x) \cap I}_{J \text{ right ideal } \subset I} = 0$.

if $J \neq 0$. $J \subset_e I$

$$\begin{tikzcd} \{a \in R \mid xya = 0\} = J' \arrow[r, phantom, "\subset_e"] \arrow[d] & R \arrow[d, "L_y"] \\ = J \arrow[r, phantom, "\subset_e"] \arrow[u, phantom, shift left=4, "\Uparrow"] & I \end{tikzcd}$$

$xyJ' = 0$.

$xy \in \mathrm{LAnn}(J')$ ess $\Rightarrow xy = 0$.

($rk = 1$ ✓)

Induction step:

$I' \subset I$

$\mathrm{grk}(I') = \mathrm{grk}(I) - 1$.

$x' \in I'$. $\mathrm{RAnn}(x') \cap I' = 0$.

If $\mathrm{RAnn}(x') \cap I = 0$ ✓

If not, $U = \mathrm{RAnn}(x') \cap I$

$$I' \oplus U \subset I$$

x' uniform.

$u \in U$.

$\mathrm{RAnn}(u) \cap U = 0$.

$x = x' + u$ satisfies

$\mathrm{RAnn}(x) \cap I = 0$. ∎