Lecture 21

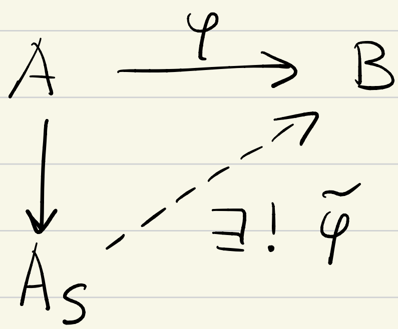
Localization

Commutative setting:

A, $S \subset A$ multiplicative set. $0 \notin S$
($1 \in S$, closed under mult).

$A \xrightarrow{i} A_S$ the localization of A wrt S.
s.t. $i(S) \subset A_S^\times$.

Universal property: $\forall$ comm ring B



s.t. $\varphi(S) \subset B^\times$

Description of A_S:

- every element has the form $i(a)\, i(s)^{-1}$
  some $a \in A$, $s \in S$.
  "a/s" "$\frac{a}{s}$"
  "as^{-1}", "$s^{-1}a$".

$$\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$$

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}.$$

an elt may be written as $\frac{a}{s}$ or $\frac{b}{t} \in A_S$.

$$\frac{a}{s} = \frac{b}{t} \iff (at - bs)s' = 0$$

for some $s' \in S$.

$$A \xrightarrow{i} A_S$$

- $\ker(i) = \{ a \in A \mid as = 0, \text{ for some } s \in S \}$.

Ex: $S = \{ f^n \mid n \geq 0 \}$. $f \in A$, non-nilp.

$$\rightsquigarrow A_S = A_f = A[f^{-1}]$$

$$= A[x]/(xf - 1).$$

Ex. $\mathfrak{p} \subset A$ prime ideal.

$S = A \setminus \mathfrak{p}$.

A_S denoted by $\underline{\underline{A_{\mathfrak{p}}}}$. ← local ring

max ideal $\mathfrak{p}A_{\mathfrak{p}}$.

residue field

$\boxed{\mathrm{Frac}(A/\mathfrak{p})}$

Ex. A = domain. $S = A \setminus 0$.

$\rightsquigarrow A_S = \mathrm{Frac}(A)$ field of fractions of A.

Non-commutative case:

$S \subset R$ any subset.

$\varphi : R \longrightarrow R'$

- $\varphi(S)$ is left invertible.
- $\varphi(S)$ is right invertible.
- $\varphi(S)$ is invertible.

<u>Thm</u> $S \subset R$ any subset.

$\exists$ $R \xrightarrow{i} R_S$ (possibly 0) ring

s.t. $\forall$

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & R' \\ {\scriptstyle i}\downarrow & \nearrow_{\exists!\, \tilde{\varphi}} & \\ R_S & & \end{array}$$

ring homo. (have 1)

$\varphi(S)$ is invertible

In other words:

If $\exists$ $R \xrightarrow{\varphi} R'$ st. $\varphi(S)$ is invertible.

Then above R_S is a usual ring ($1 \neq 0$).

If $\nexists$ ———— " ————.

$R_S = 0$.

<u>Rk.</u> $\exists$ similar version for "$\varphi(s)$ equipped with left inverses"

" ———— right ————"

$R \xrightarrow{\varphi} R'$

$\forall s \in S$. $\sigma(s) \in R'$. st. $\sigma(s) \cdot \varphi(s) = 1$.

$\underline{\text{Ex}}$. $R = k\langle x, y\rangle$

$S = \{x\}$.

$$\boxed{k\langle x,y,\xi\rangle / (x\xi - 1)}$$

$$R_S = \underset{\sim\sim\sim}{k\langle x, y, \xi\rangle} \Big/ (\underset{\sim}{x\xi - 1}, \underset{\sim}{\xi x - 1}).$$

$k\langle x,y,\xi\rangle$ has k-basis: words in x, y, ξ.

when x and ξ are next to each other can cancel them.

$$(\overset{w_1}{\cdots}) \, x\xi \, (\overset{w_2}{\cdots}) = w_1 w_2.$$

R_S has a k-basis : words in x, y, ξ where x and ξ are not next to each other.

$$\xi = x^{-1}$$

word in R_S $\quad \boxed{x^{a_1} y^{b_1} x^{a_2} y^{b_2} \cdots y^{b_n} x^{a_{n+1}}}$

$b_i \geq 0$.

$\boxed{a_i \in \mathbb{Z}}$ $\qquad x^{-3} = \xi^3$.

$$R_S = \underline{k\langle x, x^{-1}, y\rangle}.$$

$k\langle x, y\rangle$ → φ s.t. $\varphi(x)$ is right invertible but not left invertible

$k\langle x, y, \xi\rangle / (x\xi - 1)$ → R

$\xi \mapsto$ a right inverse of $\varphi(x)$

$\downarrow$ not an isom.

$k\langle x, y, \xi\rangle / (x\xi - 1, \xi x - 1)$

$R = \mathrm{End}_k(V)$, $\dim_k V = \infty$.

$\varphi(x)$ surj but not injective

$\varphi(y)$ any.

k-basis: ξx

$y^{?} \cdots y^{?} \xi^{a} x^{b} y^{?} \cdots y^{?} \cdots y^{?}$

Pf (Thm) Recall tensor-algebra.

$R \circlearrowright M \circlearrowleft R$.

$$T_R(M) = R \oplus M \oplus M \otimes_R M \oplus \cdots$$

$$M = \left(R \otimes_{\mathbb{Z}} R\right)^{\oplus S}.$$

$(1 \otimes 1)_{s \in S} = s^*$

$R_{\mathbb{Z}}\langle s ; s \in S\rangle$.

$\|$

$T_R(M) / \left(\begin{matrix} s s^* - 1 \\ s^* s - 1 \end{matrix}, \ \forall s \in S \right)$ ← may contain 1.

Univ. prop.

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & R' \\ \downarrow & \nearrow & \\ T_R(M) \Big/ \left(\begin{array}{c} s s^* - 1 \\ s^* s - 1 \end{array} \right) & & \end{array}$$

$\varphi(S)$ invertible

$\tilde{\varphi}(s^*) = \varphi(s)^{-1}$.

∎

Ore conditions.

multiplicative

$S \subset R$, Suppose $\exists$ $R \xrightarrow{\varphi} R'$ s.t.

① $\varphi(S)$ invertible.

② any elt in R' has the form $\varphi(a)\varphi(s)^{-1}$ (right fraction)
some $a \in R$, $s \in S$

③. $\ker(\varphi) = \{ a \in R \mid a \cdot s = 0, \text{ for some } s \in S \}$.

$\Longrightarrow$ properties of S.

$\forall$ $s \in S$, $a \in R$.

$\varphi(s)^{-1} \cdot \varphi(a) \in R'$.

② $\Rightarrow$ $\exists$ $t \in S$, $b \in R$. st.

$\varphi(s)^{-1}\varphi(a) = \varphi(b)\varphi(t)^{-1}$.

$\Longleftrightarrow \quad \varphi(a)\varphi(t) = \varphi(s)\varphi(b)$

$\varphi(at - sb) = 0$

③ $\Rightarrow \exists\ s' \in S.$

$(at-sb)s' = 0.$

i.e., $a\underbrace{(ts')}_{\in S} = s\underbrace{(bs')}_{\in R} \in R.$

Conclusion: $\forall\ a \in R,\ s \in S.$

$\exists\ b \in R,\ t \in S$ s.t.

$a\,t = s\,b$

i.e., $\boxed{aS \cap sR \neq \phi}$ right permutable.

"$s^{-1}a = bt^{-1}$"

Next. Suppose $a \in R$.

s.t. $sa = 0$. (some $s \in S$)

then $a \cdot s' = 0$ for some $s' \in S$.

right reversible

($sa = 0$. $\varphi(s)\varphi(a) = 0 \in R'$. $\Rightarrow \varphi(a) = 0$.

($\varphi(s)$ invertible.)

③ $\Rightarrow as' = 0$.)

Def. $S \subset R$ mult subset is <u>right Ore set</u> if it is right permutable and right reversible.

Thm. Let $S \subset R$ be a right Ore set. Then $R \xrightarrow{i} R_S$ has properties

0) $i(S)$ is invertible.
1) every elt in R_S has the form $i(a)\,i(s)^{-1}$ (some $a \in R$, $s \in S$)
2) $\ker(i) = \{ a \in R \mid as = 0 \text{ some } s \in S \}$.

Pf. First define R' as a left R-mod.

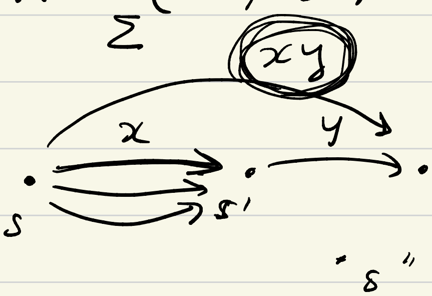
$$\boxed{R \cdot s^{-1}} \quad \text{free } R\text{-mod of rk } 1.$$

$$R' := \varinjlim_{\Sigma} R \cdot s^{-1} \qquad \text{(colimit)}.$$

Σ is the category:

$$\mathrm{Obj}(\Sigma) = S.$$

$$\mathrm{Hom}_\Sigma(s, s') = \{ x \in R \mid sx = s' \}.$$



$$M : \Sigma \longrightarrow R\text{-mod}.$$

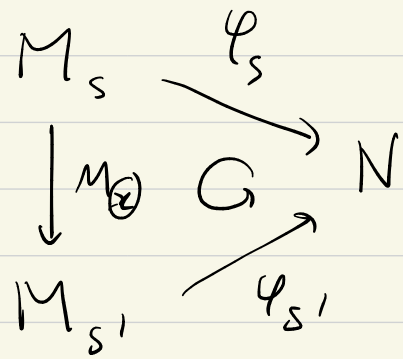
$$s \longmapsto M_s.$$

$$(s \xrightarrow{x} s') \longmapsto (M_s \xrightarrow{M_x} M_{s'})$$

$$M_s \longrightarrow \varinjlim_{\Sigma} M \quad \text{an } R\text{-mod}.$$

$$M_x : M_s \to M_{s'}, \quad M_{s'} \longrightarrow \varinjlim_{\Sigma} M \quad (\forall s \in S)$$

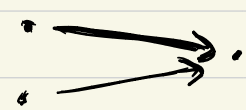
Any R-mod N.



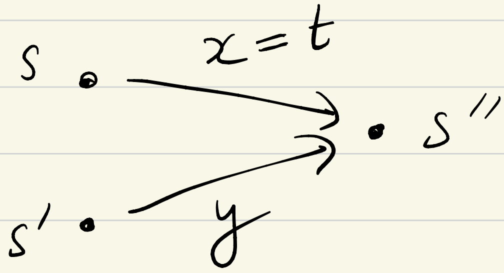
Then $\exists!$ $\varinjlim_{\Sigma} M \xrightarrow{\varphi} N$

s.t. $\forall$ $M_s \longrightarrow \varinjlim M \xrightarrow{\varphi} N$.

Direct limit: Σ = poset. filtered.

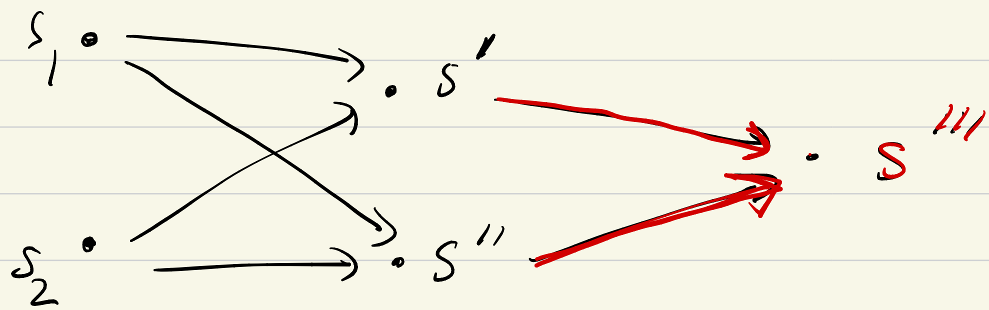


Special property of Σ.

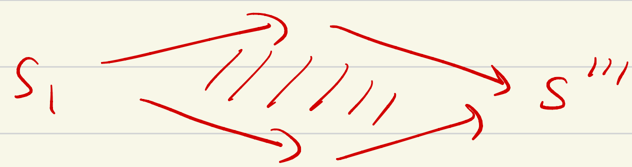
① 

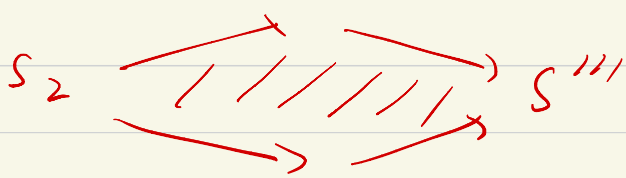
$$sS \cap s'R \neq \emptyset.$$

$$s'' = st = s' \cdot y, \quad st \in S, \ t \in S, \ y \in R.$$

② 

s.t.





$$R' = \varinjlim_{\Sigma} \left(R \cdot s^{-1} \right), \quad R \cdot s^{-1} = M_s$$

$s \xrightarrow{x} s'$

$$
\begin{array}{ccc}
M_s & \longrightarrow & M_{s'} \\
\| & & \| \\
R \cdot s^{-1} & & R \cdot (s')^{-1} \\
\| & & \| \\
R & \xrightarrow{(-)x} & R
\end{array}
$$

$$a s^{-1} \qquad\qquad (\quad)(s')^{-1}$$

$$= (\quad)(sx)^{-1}$$

$$= (ax)\, x^{-1} \cdot s^{-1}$$