Lecture 18 11/4

Central simple algebras

Brauer group

commutative subalgebras

Thm : R/k c.s.a.

S : k-alg.

$$\{\text{ideals in } S\} \longleftrightarrow \{\text{ideals in } R \otimes_k S\}$$

$$I \longmapsto R \otimes_k I$$

$$J \cap (1 \otimes S) \longleftarrow J$$

Special case: $R = M_n(k)$.

Thm $\Rightarrow$ ideals in $M_n(S)$ are of the form $M_n(I)$ $(I \subset S)$.

Pf: $I \longmapsto R \otimes_k I \longmapsto (R \otimes I) \cap (1 \otimes S) = I$.

Need to show: $J \longmapsto I := J \cap (1 \otimes S)$

$R \underset{k}{\otimes} I \subset J$.

is an equality.

replace S by S/I

J by $J \big/ R \otimes I \subset R \otimes (S/I)$.

we may assume $I = 0$.

i.e., $J \cap (1 \otimes S) = 0$.

want to show $J = 0$.

$\{r_i\}$ k-basis for R.

$$\boxed{\sum_{i=1}^{n} r_i \otimes s_i} \in J. \qquad s_i \in S.$$

$$\sum r' \cdot r_i \cdot r'' \otimes s_i \in J. \qquad r', r'' \in R.$$

Recall (density applied to

$$R \underset{k}{\otimes} R^{op} \circlearrowright \underbrace{R}$$

simple $R \otimes R^{op}$-mod.

$\Longrightarrow$ $R \underset{k}{\otimes} R^{op} \longrightarrow \mathrm{End}_D(R)$ has dense image.

$\left(D = \mathrm{End}_{R \underset{k}{\otimes} R^{op}}(R). \right)$

$$Z(R) = k.$$

$$Z(R) = \mathrm{End}_{R \otimes_k R^{op}}(R) = D$$

$R \subset R \circlearrowleft x$

$\Rightarrow \quad D = k.$

hence $\quad R \otimes_k R^{op} \longrightarrow \mathrm{End}_k(R)$

has dense image.

i.e., $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \xrightarrow{\exists\, a \in R \otimes_k R^{op}} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

(k-linearly indep $\in R$) $\qquad$ (in R)

Apply this

$$\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} \xrightarrow{\exists\, a \in R \otimes_k R^{op}} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\underbrace{\boxed{a \cdot r_i}}_{\sum \text{ left and right mult on } r_i.} \longmapsto \delta_{i1}.$$

$$\sum_{i=1}^{n} (a \cdot r_i) \otimes s_i \in J.$$

$$\underset{\substack{\| \\ 1 \otimes s_1}}{}$$

by assumption $J \cap (1 \otimes S) = 0. \Rightarrow s_1 = 0.$

Similarly. all $s_i = 0$. $\Longrightarrow J = 0$. ☒

<u>Prop</u>. R/k c.s.a. $\dim_k R < \infty$.
then $\dim_k R = n^2$.

<u>Pf</u>. $R \otimes_k \bar{k}$ c.s.a. / $\bar{k}$, $\dim_{\bar{k}} (R \otimes \bar{k}) < \infty$.

$$\Longrightarrow R \otimes_k \bar{k} = M_n(\underbrace{D}_{\text{div. alg} / \bar{k}})$$

$D = \bar{k}$.

$$\| \\ M_n(\bar{k}).$$

$$\dim_k (R) = \dim_{\bar{k}} (R \otimes \bar{k}) = n^2.$$

In particular. D/k is a finite-dim'l division algebra.

$Z(D)$: a finite extn of k.

$$\dim_{Z(D)} D = n^2.$$

$$\dim_k D = n^2 \cdot [Z(D) : k].$$

Notation: R is a f.d. alg / k

$$[R : k] = \dim_k R.$$

<u>Prop.</u> R/k c.s.a. $[R:k] < \infty$.

Then $\varphi : R \otimes_k R^{op} \xrightarrow{\sim} \mathrm{End}_k(R) = M_{n^2}(k)$

$n^2 = [R:k]$.

<u>Pf.</u> Density $\Rightarrow \varphi$ surjective.

$(R \otimes_k R^{op}) / k$ is c.s.a.

$\Rightarrow R \otimes_k R^{op}$ is simple. $\Rightarrow \varphi$ is injective.

<u>Brauer group</u> of k.

$$Br(k) = \{ D/k : \overset{\text{f.d.}}{\text{central}} \text{ division alg} \} / \cong$$

$$= \{ R/k : \overset{\text{f.d.}}{\text{c.s.a.}} \} / \sim$$

$$\underset{M_{n_1}(D_1)}{\overset{\|}{R_1}} \sim \underset{M_{n_2}(D_2)}{\overset{\|}{R_2}} \iff D_1 \cong D_2$$

(k-linear)

$$D \sim M_2(D) \sim M_3(D) \sim \cdots$$

group mult. $[R] \in Br(k)$.

$$[R_1] \cdot [R_2] = [R_1 \otimes_k R_2]$$

unit $[k]$.

<u>inverse</u>,

$$[R]^{-1} = [R^{op}]$$

$\because$ $[R]\cdot[R^{op}] = [R \otimes_k R^{op}]$

$= [M_{n^2}(k)]$ $\quad n^2 = [R:k]$

$= [k] = \text{unit}$.

commutative. ✓

$Br(A)$.

c.s.a./k ⇝ Azumaya algebra over A.

(R/A Azumaya alg.

$A \to k$.

$R \otimes_A k/k$ is c.s.a.).

$$Br(\mathbb{F}_q) = \{1\}.$$

$$Br(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z} = \{\mathbb{R}, \mathbb{H}\}$$

$\mathbb{R} \mapsto 0$, $\mathbb{H} \mapsto 1$

$\mathbb{H}^{op} \simeq \mathbb{H}$.

$i \mapsto i$

$j \mapsto j$

$k \mapsto -k$

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}^{op} \simeq M_4(\mathbb{R})$$

$\mathbb{Q}_p$.

$$\boxed{\mathrm{Br}(\mathbb{Q}_p) \simeq \mathbb{Q}/\mathbb{Z}.}$$

$$D_{a/b} \leftsquigarrow 0 \leq \frac{a}{b} < 1.$$

f.d. central div. $/\mathbb{Q}_p$.

$$[D : \mathbb{Q}_p] = b^2.$$

$$[D_{a/b} \otimes_{\mathbb{Q}_p} D_{c/d}] = [D_{\frac{a}{b}+\frac{c}{d}}]$$

$\frac{ad+dc}{bd} = \frac{e}{f}$.

~~dim~~ dim over $\mathbb{Q}_p$

$b^2 d^2$ $\qquad$ f^2

bd/f

$$D_{a/b} \otimes_{\mathbb{Q}_p} D_{c/d} \simeq M_{bd/f}(D_{\frac{e}{f}}).$$

HW: $\mathbb{Q}_p$, $\underline{\mathbb{F}_p((t))}$.

$$\mathrm{Br}(\mathbb{F}_p((t))) \simeq \mathbb{Q}/\mathbb{Z}$$

$$D_{a/b} \longleftrightarrow \frac{a}{b}$$

is a cyclic algebra

$$\mathbb{F}_{p^b}((t))\langle x ; \mathrm{Frob}\rangle / (x^b - t^a).$$

$Br(\mathbb{Q})$.

$$D_1 \overset{\text{Morita}}{\sim} D_2 \iff \boxed{D_1 \cong D_2.}$$

$$D_1\text{-v.s.} \cong D_2\text{-v.s.}$$

$$D_1 \longmapsto D_2$$

$$\begin{array}{ccc} \mathrm{End}_{D_1}(D_1) & \cong & \mathrm{End}_{D_2}(D_2). \\ \wr\! \mid & & \wr\! \mid \\ D_1^{op} & & D_2^{op} \end{array}$$

$$D_1 \overset{k\text{-linear Morita}}{\sim} D_2 \iff D_1 \overset{k\text{-linear.}}{\cong} D_2$$

$$k \longrightarrow L.$$

$$\begin{array}{ccc} Br(k) & \longrightarrow & Br(L). \\ \in & & \in \\ [R] & \longmapsto & [R \otimes_k L] \end{array}$$

$$Br(\mathbb{Q}) \longrightarrow \left(\prod_{p \text{ prime}} Br(\mathbb{Q}_p) \right) \times Br(\mathbb{R}).$$

<u>Fact</u>: 1) image lies in $\left(\bigoplus_p Br(\mathbb{Q}_p) \right) \oplus Br(\mathbb{R})$.

2) it is injective.

3). image is

$$\ker\left(\left(\bigoplus_p \mathrm{Br}(\mathbb{Q}_p)\right) \oplus \underset{\overset{\cong}{\frac{1}{2}\mathbb{Z}/\mathbb{Z}}}{\mathrm{Br}(\mathbb{R})} \xrightarrow{\text{sum}} \mathbb{Q}/\mathbb{Z} \right)$$

(class field theory).

e.g. quaternion algebras $D/\mathbb{Q}$.
(i.e. $[D:\mathbb{Q}]=4$).

$$\{\text{quaternion alg } D/\mathbb{Q}\}/\cong$$

$$\updownarrow$$

$$\ker\left(\left(\bigoplus_p \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}\right) \oplus \underbrace{\tfrac{1}{2}\mathbb{Z}/\mathbb{Z}}_{\mathrm{Br}(\mathbb{R})} \xrightarrow{\text{sum}} \tfrac{1}{2}\mathbb{Z}/\mathbb{Z} \right)$$

$$\cong$$

$$\mathrm{Fun}_{\text{fin}}(\underset{\underset{\text{primes}}{\parallel}}{\mathcal{P}}, \mathbb{Z}/2) \longleftrightarrow \text{subsets of } \mathcal{P}.$$

e.g. $\mathbb{H}/\mathbb{Q}$ $1, i, j, k$. usual mult.

$$\longleftrightarrow \{2, \mathbb{R}\}.$$

i.e. $\mathbb{H} \otimes_{\mathbb{Q}} \mathbb{Q}_2$ and $\mathbb{H} \otimes_{\mathbb{Q}} \mathbb{R}$ are nontriv. div.

$p \neq 2 \qquad \mathbb{H} \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq M_2(\mathbb{Q}_p).$

Commutative subfields.

<u>Prop.</u> R/k. c.s.a. $[R:k] = n^2$.
$k \subset L \subset R$. subfield.

$\Rightarrow \quad [L:k] \mid n.$

If L is a maximal comm subalg of R,
then $[L:k] = n$.

<u>Ex</u>. $R = M_n(k) = \mathrm{End}_k(V)$. $\dim_k V = n$.

$\underset{\text{field}}{L} \subset R = \mathrm{End}_k(V)$

$k \subset L \circlearrowright V$

V is L-v.s.

$\Rightarrow \quad n = \dim_k V = \dim_L V \cdot [L:k].$

$[L:k] \mid n.$

<u>Ex</u>. $R = D$ central div. $/k$. $[D:k] = n^2$.
$k \subset L \subset D$.
$L \subset L_0 =$ max comm subalgebra
L_0 is a field.
$\forall x \in L_0 \setminus 0. \quad x^n + \cdots + c_0 = 0 \qquad c_i \in k.$
irred. $\in k[x]$. $\quad c_0 \neq 0 \Rightarrow x$ invertible

$$[L_0 : k] = n$$

$$[L : k] \mid [L_0 : k] = n.$$

Pf (prop.) $L \subset R \circlearrowright R \circlearrowleft R$

$$\mathrm{End}_L(R) = \{ T : R \to R \text{ comm with } L\text{-mult on the left} \}$$

$$\begin{array}{ccc} R \otimes_k R^{op} & \xrightarrow{\sim} & \mathrm{End}_k(R). \\ \cup & & \cup \\ \text{centralizer of } L \otimes 1 & \xrightarrow{\sim} & \mathrm{End}_L(R). \\ \| & & \\ Z_R(L) \otimes_k R^{op} & & \end{array}$$

$Z_R(L)$: centralizer of L in R.

Compute $\dim_k(-)$ of both sides.

$$[Z_R(L) : k]\,[R : k] = \underbrace{[R : L]^2 \cdot [L : k]}_{\dim_L R \,=\, [R:k]\cdot[R:L]}.$$

$$\Rightarrow \quad [Z_R(L) : k] = [R : L].$$

$$[Z_R(L) : k] = [Z_R(L) : L] \cdot [L : k]$$

$$\underbrace{k \subset L \subset Z_R(L) \subset R}_{n^2}$$

$$\cancel{[R : Z_R(L)]}$$

$$\frac{[R:k]}{[Z_R(L):k]}$$

multiply both sides by $[L:k]$ $\Longrightarrow$

$$[L:k]^2 \cdot [Z_R(L):L] = [R:k] = n^2$$

$$\Longrightarrow \qquad [L:k] \mid n$$

If L is max comm. $Z_R(L) = L$,

$$\Longrightarrow \qquad [L:k]^2 = n^2$$

$$[L:k] = n.$$

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